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AN EXTENSION OF MAMIKON'S THEOREM

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Abstract: Mamikon A. Mnatsakanian showed the area swept out by a tangent is equal to the area of its corresponding Ikon (collecting all the tangent lines as if the originated from one common point). Here we extend Mamikon's theorem to areas swept by normal lines and other regions.

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Key Words: Mamikon's Theorem, tangent sweeps, ikon

1. Introduction

Mamikon A. Mnatsakanian showed Theorem 1.1 in [1] a beautiful and simple piece of geometric intuition. It is known that areas of sections of a tangent developable are independent of three dimensions, Mamikon showed these areas are independent of arclength. We will show an extension to this theorem.

Let $\boldsymbol{\gamma}$ be a regular unit parametrized curve in $\mathbb{R}^2$. Let $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{T}(u)$ be a parametrization of the developable tangent surface of $\boldsymbol{\gamma}$. Let region R of $\boldsymbol{\sigma}(u,v)$ be given by $u_1 \leq u \leq u_2$ and $0 \leq u \leq v(u)$. The region R is a developable normal surface. Let ϕ be the angle made by $\boldsymbol{T}$ and some fixed vector (say the x-axis). We denote the **area of the ikon** as AI

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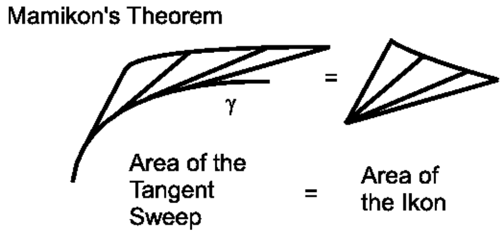

Figure 1

$$AI = \int v(\phi)\, d\phi.$$

where $v(\phi) = v(s(\phi))$ is now independent of arclength. Mamikon then states

Theorem 1.1 (Mamikon's Theorem)**.** *The area of the region on a developable tangent surface equals the area of the ikon.*

That is the area of the tangent sweep equals the area of the ikon, see Figure 1.

2. A First Extension

We will extend these results to a new structure. Let $\boldsymbol{\gamma}$ be a regular unit parametrized curve in $\mathbb{R}^2$. We define the surface $\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{N}(u)$ to be a **normal developable**. Let region R of $\boldsymbol{\sigma}(u, v)$ be given by $u_1 \leq u \leq u_2$ and $v_1(u) \leq u \leq v_1(u)$. We now define the notions of **area of the ikon** (again denoted AI) and the **perpendicular area** denoted as PA for a developable normal surface. Let

$$AI = \int v(\phi)\, d\phi \quad \text{and} \quad PA = \iint_R 1\, dudv.$$

where $v(\phi) = v(s(\phi))$ is now independent of arclength. Notice $v(\phi)$ is the length of the normal whereas in Mamikon's Theorem $v(\phi)$ was the length of the tangent line.

The following is a Corollary of Theorem 3.1.

Corollary 2.1. *The area of the surface $\boldsymbol{\sigma}(u, v)$ over the region R is $PA + AI$ where the region $R = \{(u, v) : u_1 \leq u \leq u_2$ and $v_1(u) \leq v \leq v_2(u)\}$.*

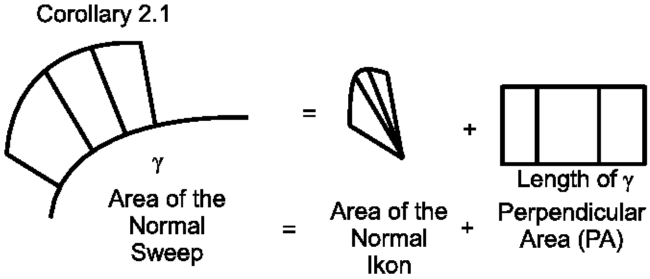

Figure 2

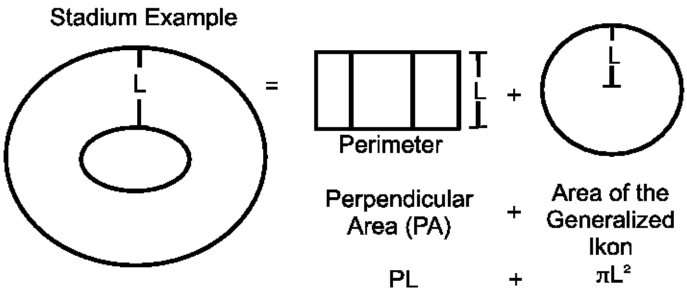

Figure 3

That is the area of the region on a developable normal surface equals the area of the ikon plus the perpendicular area, see Figure 2.

An Application - An Elliptical Gasket (or a buffered stadium): Let a surface be given by $\boldsymbol{\sigma}(u,v) = \langle a\cos(u), b\sin(u)\rangle + v\langle\cos(u), \sin(u)\rangle$. Let the region R of the surface $\boldsymbol{\sigma}(u,v)$ be given by $0 \le u \le 2\pi$ and $0 \le v \le \ell$ then the surface area of the region is easily seen to be $SA = PA + AI = 2\pi\sqrt{ab}\ell + \pi\ell^2$.

More generally, if we only know the perimeter (P) of the stadium (or any convex body with a smooth boundary) and we extend outward at a constant length, L, normal to the stadium our area of our normal sweep is $PA + AI = PL + \pi\ell^2$, see Figure 3.

3. Further Extension

What regions that are partially like a normal developable and partial a tangent developable? Let γ be a regular unit parametrized curve in $\mathbb{R}^2$. Let

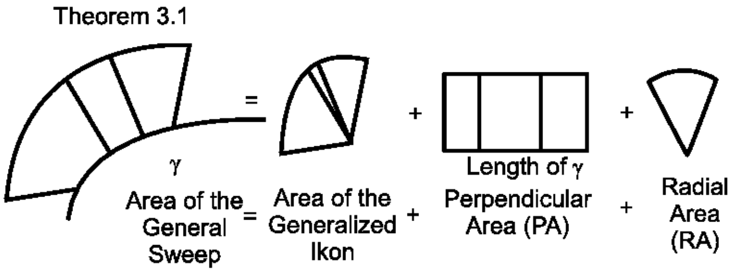

Figure 4

$\theta : [u_1, u_2] \to [0, \pi/2]$ be a once differentiable and decreasing function. Also let $\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v[\cos(\theta(u))\boldsymbol{N}(u) + \sin(\theta(u)T(u)]$ be a surface parametrization. We now have a surface parametrized by $\boldsymbol{\sigma}(u, v)$ which is a hybrid of the tangent and normal developables.

We will use the same definition for AI. Again note that the AI could have a different geometric interpretation then in either Theorem 1.1 or in Corollary 2.1. For the PA we will have the exact same geometric interpretation, but with a different formula

$$PA = \int_{u_1}^{u_2} v \sin(\theta(u))\, du.$$

We will need one further idea, **Radial Area**, denoted RA. The RA is the area over our angle θ changes over the length v, with formula

$$RA = \int_{\theta_1}^{\theta_2} \frac{1}{2} v^2(\theta)\, d\theta \text{ (where } \theta(u_1) = \theta_1 \text{ and } \theta(u_2) = \theta_2).$$

Theorem 3.1. *Then the area of the surface $\boldsymbol{\sigma}(u, v)$ over the region R is $PA + AI + RA$ where the region $R = \{(u, v) : u_1 \le u \le u_2$ and $v_1(u) \le v \le v_2(u)\}$.*

Or alternatively the Theorem 3.1 can be seen in Figure 4.

Proof. Let $\boldsymbol{\sigma}(u, v) = \gamma(u) + v[\sin(\theta(u))\boldsymbol{N}(u) + \cos(\theta(u))\boldsymbol{T}(u)]$ be the parametrization of the surface. Now to compute the first fundamental form we have

$$
\begin{aligned}
\boldsymbol{\sigma}_u(u,v) &= \dot{\gamma} + v\left[\sin(\theta)\dot{\boldsymbol{N}}(u) + \cos(\theta)\dot{\theta}\boldsymbol{N}(u) + \cos(\theta)\dot{\boldsymbol{T}}(u) - \sin(\theta)\dot{\theta}\boldsymbol{T}(u)\right] \\
&= \boldsymbol{T} + v\left[\sin(\theta)(-\kappa\boldsymbol{T}) + \cos(\theta)\dot{\theta}\boldsymbol{N}(u) + \cos(\theta)(\kappa\boldsymbol{N}) - \sin(\theta)\dot{\theta}\boldsymbol{T}(u)\right] \\
&= (1 - v\sin(\theta)(\kappa + \dot{\theta}))\boldsymbol{T} + v\cos(\theta)(\dot{\theta} + \kappa)\boldsymbol{N}(u)
\end{aligned} \tag{1}
$$

using the Frenet-Serret equations in equation (1) (found in any elementary differential geometry book such as [2]). And we have $\boldsymbol{\sigma}_v(u,v) = \sin(\theta)\boldsymbol{N}(u) + \cos(\theta)\boldsymbol{T}(u)$. So

$$
\begin{aligned}
E = \boldsymbol{\sigma}_u(u,v)\cdot\boldsymbol{\sigma}_u(u,v) &= \left[1 - v\sin(\theta)(\kappa + \dot{\theta})\right]^2 + v^2\cos^2(\theta)(\dot{\theta} + \kappa)^2 \\
&= 1 - 2v\sin(\theta)(\dot{\theta} + \kappa) + v^2(\dot{\theta} + \kappa)^2 \\
F = \boldsymbol{\sigma}_u(u,v)\cdot\boldsymbol{\sigma}_v(u,v) &= \sin(\theta)v\cos(\theta)(\dot{\theta} + \kappa) + \cos(\theta)(1 - v\sin(\theta)(\kappa + \dot{\theta})) \\
&= \cos(\theta), \text{ and} \\
G = \boldsymbol{\sigma}_v(u,v)\cdot\boldsymbol{\sigma}_v(u,v) &= 1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sqrt{EG - F^2} &= \sqrt{(1 - 2v\sin(\theta)(\dot{\theta} + \kappa) + v^2(\dot{\theta} + \kappa)^2) - \cos^2(\theta)} \\
&= \sqrt{\sin^2(\theta) - 2v\sin(\theta)(\dot{\theta} + \kappa) + v^2(\dot{\theta} + \kappa)^2} \\
&= \sqrt{\left(\sin(\theta) - v(\dot{\theta} + \kappa)\right)^2} = \sin(\theta) - v(\dot{\theta} + \kappa).
\end{aligned}
$$

Thus the area as given by the second fundamental form is

$$
\begin{aligned}
&\iint_R \sqrt{EG - F^2}\, du\, dv \\
&\qquad = \int_{u_1}^{u_2}\int_{v_1(u)}^{v_2(u)} \sin(\theta) - v(\dot{\theta} + \kappa)\, dv\, du \\
&\qquad = \int_{u_1}^{u_2}\int_{v_1(u)}^{v_2(u)} \sin(\theta)\, dv\, du - \int_{u_1}^{u_2}\int_{v_1(u)}^{v_2(u)} v\kappa\, dv\, du \\
&\qquad\quad - \int_{u_1}^{u_2}\int_{v_1(u)}^{v_2(u)} v\dot{\theta}\, dv\, du \\
&\qquad = PA - \int_{u_1}^{u_2} \frac{1}{2}v^2(u)\kappa\, du - \int_{u_1}^{u_2} \frac{1}{2}v^2(u)\dot{\theta}\, du
\end{aligned}
$$

$$
\begin{aligned}
&= PA + \int_{u_1}^{u_2} \frac{1}{2} v^2(\phi)\, d\phi + \int_{u_1}^{u_2} \frac{1}{2} v^2(\theta)\, d\theta \\
&= PA + AI + RA. \qquad \square
\end{aligned}
$$

References

[1] Mamikon A. Mnatsakanian, Areas on Developable Surfaces, *Proceedings of the Armenian Academy of Sciences*, **Volume 73**, No. 2 (1981), 97-102.

[2] Pressley, Andrew, *Elementary Differential Geometry*, Springer, USA (2000).