SOME EQUIVALENT NORMS ON THE HILBERT SPACE

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ABSTRACT. We present a family of some new Tsirelson-type norms for the separable Hilbert space. Our results extend some results of S. Bellenot, J. Bernués and I. Deliyanni and provide candidates for distorted norms on the Hilbert space.

In this note we present a family of some new Tsirelson-type norms for the separable Hilbert space ℓ_2. The motivation for presenting these norms is the following question of E. Odell and Th. Schlumprecht [1] which is also mentioned by T.W. Gowers [5]:

Question 1. *Is it possible, for $\lambda > 0$ to explicitly define an equivalent norm $|\cdot|$ on ℓ_2 such that every infinite dimensional subspace Y of ℓ_2 contains two vectors y_1 and y_2 with $\|y_1\|_2 = \|y_2\|_2 = 1$ (where $\|\cdot\|_2$ denotes the usual norm of ℓ_2) and $|y_1|/|y_2| > \lambda$?*

An implicitly defined norm with the above property exists by the solution of the famous distortion problem by Odell and Schlumprecht [1, 2]. The family of norms that we present gives candidates for the solution of Question 1. Some of the norms of our family were first presented by S. Bellenot [3] which recently A.M. Pelczar [6] proved that these norms do not answer Question 1. Another purpose of present note, is to extend some results of Bellenot [3], J. Bernués and I. Deliyanni [4].

In order to define the new norms on ℓ_2 we first introduce some notation. For $x = (x(i)) \in \ell_2$ and $E \subseteq \mathbb{N}$ we denote by Ex the natural projection of x on E, i.e. $Ex = ((Ex)(i))$ where $(Ex)(i) = x(i)$ for all $i \in E$ and $(Ex)(i) = 0$ otherwise. Let c_{00} be the vector space of scalar sequences with finite support. Set

$$\mathcal{A} = \left\{ y = (y(1), y(2), \ldots, y(N)) \in (0, \infty)^N : N \in \mathbb{N}, \ N \geq 2, \ \sum_{i=1}^N |y(i)|^2 = 1 \right\}.$$

We define a sequence of equivalent norms $\|\cdot\|_y$, for $y \in \mathcal{A}$, on ℓ_2 as follows. Fix $y = (y(i))_{i=1}^N \in \mathcal{A}$. Then $\|\cdot\|_y$ is the unique norm which satisfies

$$
(1) \qquad \|x\|_y = \sup \sum_{i=1}^N y(i) \|E_i x\|_y \vee \|x\|_\infty \text{ for every } x \in c_{00},
$$

where the supremum is taken with respect to any sequence of sets $E_1 < E_2 < \cdots < E_N$. Notice that in the definition of $\|\cdot\|_y$ we allow the sets E_i to be empty and we adopt the convention that "$A < \emptyset$" and "$\emptyset < B$" are valid for any $A, B \subseteq \mathbb{N}$. It is standard to show that for every $y \in \mathcal{A}$ there exists a unique norm $\|\cdot\|_y$ which satisfies (1). We provide two equivalent definitions of the $\|\cdot\|_y$. Firstly, for every fixed $y \in \mathcal{A}$ we define a sequence of

norms $\|\cdot\|_y^{(n)}$ for $n \in \mathbb{N} \cup \{0\}$ which increases to the norm $\|\cdot\|_y$, as follows. Let $\|\cdot\|_y^{(0)} = \|\cdot\|_\infty$ (the ℓ_∞ norm). If for $n \in \mathbb{N} \cup \{0\}$ the $\|\cdot\|_y^{(n)}$ has been defined, then we define

$$\text{(2)} \qquad \|x\|_y^{(n+1)} = \sup \sum_{i=1}^N y(i) \|E_i x\|_y^{(n)} \vee \|x\|_y^{(n)} \text{ for } x \in c_{00},$$

where the supremum is taken with respect to any sequence of sets $E_1 < E_2 < \cdots < E_N$. Then $\|\cdot\|_y$ is the (pointwise) limit of $\|\cdot\|_y^{(n)}$ for $n \in \mathbb{N} \cup \{0\}$, and it satisfies (1).

Secondly, we define a sequence of subsets K_n, $n = 0, 1, 2, \ldots$ of c_{00} as follows. Let

$$K_0 = \{\lambda e_i : i \in \mathbb{N}, |\lambda| = 1\}.$$

If K_n has been defined for some $n \in \mathbb{N} \cup \{0\}$ then

$$K_{n+1} = K_n \cup \{\sum_{i=1}^N y(i) x_i : x_1 < x_2 < \cdots < x_N \text{ and } x_i \in K_n \text{ for all } i \in \{1, \ldots, N\}\}.$$

Let $K = \cup_{n=1}^\infty K_n$. Then, for every $x = (x(i)) \in c_{00}$ we have that

$$\|x\|_y = \sup_{z = (z(i)) \in K} \sum_{i=1}^\infty x(i) z(i).$$

By using the Cauchy-Schwartz inequality it is easy to show by induction on $n \in N \cup \{0\}$ that $\|x\|_y^{(n)} \leq \|x\|_2$ for every $x \in c_{00}$, or $\|z\|_2 \leq 1$ for all $z \in K_n$. Thus we have that $\|x\|_y \leq \|x\|_2$ for every $x \in c_{00}$. We will also show in Theorem 3 that $\|\cdot\|_y$ is in fact equivalent to $\|\cdot\|_2$.

For the proof of our main result we will need the notions of the N-tree, (where N is an integer larger than 1), the *tree decomposition of an interval* and the *tree decomposition of a function*.

Definition 2. (a) *Let N be an integer larger than 1. An N-tree is a subset of $\{\emptyset\} \cup \cup_{k=1}^\infty \{1, 2, \ldots, N\}^k$ endowed with an order $\prec$, satisfying the following:*

(i) *$\emptyset \in \mathcal{T}$ and $\emptyset \prec t$ for all $t \in \mathcal{T} \setminus \{\emptyset\}$.*

(ii) *If $1 \leq k \leq \ell$, $n_i \in \{1, \ldots, N\}$ for $1 \leq i \leq \ell$ and $(n_1, \ldots, n_\ell) \in \mathcal{T}$ then $(n_1, \ldots, n_k) \in \mathcal{T}$.*

(iii) *If $k, \ell \in \mathbb{N}$ and $n_i, m_i \in \{1, \ldots, N\}$ then $(n_1, \ldots, n_k) \prec (m_1, \ldots, m_\ell)$ if and only if $k < \ell$ and $m_i = n_i$ for $1 \leq i \leq k$.*

A tree may be finite (respectively infinite) if it has finitely (respectively infinitely) many nodes. A node $t \in \mathcal{T}$ will be called maximal if there is no $s \in \mathcal{T}$ with $t \prec s$. If $t \in \mathcal{T} \setminus \{\emptyset\}$ we denote by t^- the unique immediate predecessor of $\mathcal{T}$. We denote by $\max(\mathcal{T})$ the set of maximal nodes of $\mathcal{T}$. If $t \in \mathcal{T} \setminus \max(\mathcal{T})$ then t^+ will denote the set of the immediate successors of t, namely $t^+ = \{s \in \mathcal{T} : t \prec s \text{ and there is no } t' \in \mathcal{T} \text{ with } t \prec t' \prec s\}$.

(b) *Let $\mathcal{T}$ be an N-tree for some $N \in \mathbb{N}$, $N > 1$. By a tree decomposition or $\mathcal{T}$-decomposition of an interval $E \subset \mathbb{R}$ we mean a family of intervals $(E_t)_{t \in \mathcal{T}}$ indexed by the tree $\mathcal{T}$, satisfying the following:*

(i) *$E_\emptyset = E$.*

(ii) *Let $1 \leq n < m \leq N$. If $(n), (m) \in \mathcal{T}$ then $E_{(n)} < E_{(m)}$. If $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \{1, \ldots, N\}$ and $(n_1, \ldots, n_k, n), (n_1, \ldots, n_k, m) \in \mathcal{T}$ then $E_{(t,n)} < E_{(t,m)}$, where for $t = (n_1, \ldots, n_k) \in \mathcal{T}$ and $n \in \{1, \ldots, N\}$ we write (t, n) to denote $(n_1, \ldots, n_k, n)$.*

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(iii) $E_t \supseteq \cup\{E_{(t,i)} : (t,i) \in \mathcal{T},\ i \in \{1, \ldots, N\}\}$.

(c) Let $\mathcal{T}$ be an N-tree for some $N \in \mathbb{N}$, $N > 1$. By a *tree decomposition* or $\mathcal{T}$-*decomposition of a function* $g : [0, \infty) \to \mathbb{R}$ *we mean a family of functions* $(g_t)_{t \in \mathcal{T}}$ *indexed by the tree* $\mathcal{T}$, *satisfying the following two properties.*

(i) $g_\emptyset = g$.

(ii) *If* $t \in \mathcal{T} \setminus max(\mathcal{T})$ *and* $t^+ = \{(t,i) : i \in F\}$ *for some* $F \subseteq \{1, \ldots N\}$ *then* $g_t = \sum_{i \in F} y(i) g_{(t,i)}$ *with* $supp\,(g_{t,i}) < supp\,(g_{t,j})$ *for every* $i, j \in F$ *with* $i < j$.

If g *has a* $\mathcal{T}$-*decomposition* $(g_t)_{t \in \mathcal{T}}$ *and* $I \subseteq max(\mathcal{T})$ *then there is a unique tree* $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$ *such that* $max(\widetilde{\mathcal{T}}) = I$. *Let* $S := \cup_{t \in I} supp\,g_t$ *and* $f := g \restriction_S$ *be the restriction of* g *on* S. *Then* f *has a* $\widetilde{\mathcal{T}}$-*tree decomposition* $(f_t)_{t \in \widetilde{\mathcal{T}}}$ *which is naturally inherited by the* $\mathcal{T}$-*decomposition of* g, *if we set* $f_t := g_t$ *for all* $t \in I$ *and* $f_t := g_t \restriction_S$ *for all* $t \in \widetilde{\mathcal{T}} \setminus max(\widetilde{\mathcal{T}})$.

Theorem 3. *For every* $y \in \mathcal{A}$, $\| \cdot \|_y$ *is equivalent to the norm of* ℓ_2.

Moreover, for $y = (y(i))_{i=1}^N \in \mathcal{A}$, *let* $a := \min_{1 \le i \le N} y(i)$, $b := \max_{1 \le i \le N} y(i)$ *and* $M \in \mathbb{N}$ *such that* $b^M \le a$. *We have that* $\| \cdot \|_y \le \| \cdot \|_2 \le (M+4)N \| \cdot \|_y$.

Proof. Fix $y = (y(i))_{i=1}^N \in \mathcal{A}$. We consider the isometric embedding $\phi : (\ell_2, \| \cdot \|_2) \to (L_2[0, \infty), \| \cdot \|_2)$ defined by $\phi(x) = \sum_{i=1}^\infty x(i) \chi_{[i-1,i)}$ where $x = (x(i)) \in \ell_2$ and χ_E denotes the characteristic function of a set E. Here, by abuse of notation, $\| \cdot \|_2$ denotes the usual Hilbert space norm on both ℓ_2 and $L_2[0, \infty)$. Let $\mathcal{B}$ denote the set of functions in $L_2[0, \infty)$ with bounded support, where for a function $f \in L_2[0, \infty)$ the support of f, $supp\,(f)$, is defined to be the essential support of f. Obviously we have that $\phi(c_{00}) \subset \mathcal{B}$. We divide the proof into three steps. In Step 1 we define a norm $| \cdot |_y$ on $\mathcal{B}$ such that the restriction $\phi \restriction_{c_{00}}$ of ϕ on c_{00}, $\phi \restriction_{c_{00}} : (c_{00}, \| \cdot \|_y) \to (\mathcal{B}, | \cdot |_y)$ is an isometric embedding. In Step 2 we define a norm $\||| \cdot \|||_y$ on $L_2[0, \infty)$ and we prove that it is equal to the usual norm $\| \cdot \|_2$ of $L_2[0, \infty)$. In Step 3 we prove that the norms $| \cdot |_y$ and $\||| \cdot \|||_y$ are equivalent. Of course, these three steps finish the proof.

Step 1: Define a norm $| \cdot |_y$ on $L_2[0, \infty)$ in terms of its dual ball, as follows. Let

$$L_0 = \{f \in L_2[0, \infty) : \|f\|_2 \le 1 \text{ and } supp\,(f) \subseteq [n-1, n) \text{ for some } n \in \mathbb{N}\}.$$

If $n \in \mathbb{N} \cup \{0\}$ and L_n has been defined, then

$$L_{n+1} = L_n \cup \{\textstyle\sum_{i=1}^N y(i) f_i : f_i \in L_n \text{ for all } i \text{ and there exists } (n_i) \subseteq \mathbb{N} \text{ such that}$$
$$supp\,(f_1) < n_1 \le supp\,(f_2) < n_2 \le supp\,(f_3) < \ldots < n_{N-1} \le supp(f_N)\}.$$

Let $L = \cup_{n=0}^\infty L_n$ and for $f \in L_2[0, \infty)$ define

$$|f|_y = \sup_{g \in L} \int_0^\infty fg.$$

Notice that $L \subset \mathcal{B}$ and for every $g \in L$ we have that $\|g\|_2 \le 1$. Thus $| \cdot |_y$ is well defined and $| \cdot |_y \le \| \cdot \|_2$. We now prove that $\phi \restriction_{c_{00}} : (c_{00}, \| \cdot \|_y) \to (\mathcal{B}, | \cdot |_y)$ is an isometric embedding. For this purpose we will need to prove (3), (4) and (5) which follow.

(3)   For every $x = (x(i)), z = (z(i)) \in c_{00}$ we have that $\displaystyle\sum_{i=1}^\infty x(i) z(i) = \int_0^\infty \phi(x) \phi(z)$.

Indeed,

$$\sum_{i=1}^{\infty} x(i)z(i) = \sum_{i=1}^{\infty} \int_{[i-1,i)} \sum_{j=1}^{\infty} x(j)\chi_{[j-1,j)} \sum_{k=1}^{\infty} z(k)\chi_{[k-1,k)}$$

$$= \sum_{i=1}^{\infty} \int_{[i-1,i)} \phi(x)\phi(z)$$

$$= \int_{0}^{\infty} \phi(x)\phi(z).$$

(4)   For all $n \in \mathbb{N} \cup \{0\}$ we have that $\phi(K_n) \subset L_n$.

In order to prove (4) we proceed by induction on n. For $n = 0$ and $z \in K_n$ we have that $z = \lambda e_k$ where $|\lambda| = 1$ and $k \in \mathbb{N}$. Thus $\phi(z) = \lambda\chi_{[k-1,k)} \in L_0$. Assume that (4) is valid for some $n \in \mathbb{N} \cup \{0\}$ and let $z \in K_{n+1}$. There exist $z_1 = (z_1(i)) < z_2 = (z_2(i)) < \cdots < z_N = (z_N(i))$ in K_n such that $z = \sum_{i=1}^{N} y(i)z_i = \sum_{i=1}^{N} y(i) \sum_{j=1}^{\infty} z_i(j)e_j$ (where (e_j) denotes the standard unit vector basis of ℓ_2). Then $\phi(z) = \sum_{i=1}^{N} y(i) \sum_{j=1}^{\infty} z_i(j)\chi_{[j-1,j)}$. Notice that for all $i \in \{1, \ldots, N\}$, $\sum_{j=1}^{\infty} z_i(j)\chi_{[j-1,j)} = \phi(z_i) \in K_n$ and

$$\sum_{j=1}^{\infty} z_1(j)\chi_{[j-1,j)} < \max(\text{supp}\,(z_1)) \le \sum_{j=1}^{\infty} z_2(j)\chi_{[j-1,j)} < \max(\text{supp}\,(z_2)) \le \cdots \le \sum_{j=1}^{\infty} z_N(j)\chi_{[j-1,j)}.$$

Thus $\phi(z) \in L_{n+1}$ which finishes the inductive proof of (4).

(5)   For every $n, m, \ell \in \mathbb{N} \cup \{0\}$, $x \in c_{00}$, with $m < \ell$ we have that
$\sup_{z \in K_n} \int_m^\ell \phi(x)\phi(z) = \sup_{g \in L_n} \int_m^\ell \phi(x)g$.

Notice that by (4) we obtain "$\le$" in (5). In order to prove "$\ge$" in (5) we use induction on n. Let $n = 0$, $g \in L_n$ and $m < \ell$ in $\mathbb{N}$. There exists $k \in \mathbb{N}$ such that $\text{supp}\,(g) \subseteq [k - 1, k)$ and $\|g\|_2 \le 1$. Thus $\int_m^\ell \phi(x)g = x(k) \int_{[m,\ell) \cap [k-1,k)} g$ since $\phi(x)$ is constant on $[k - 1, k)$. The last expression is equal to zero if $[m, \ell) \cap [k - 1, k) = \emptyset$. If $[m, \ell) \cap [k - 1, k) \ne \emptyset$ then $[k - 1, k) \subseteq [m, \ell)$ and the last expression is dominated by

$$|x(k)| \int_{[k-1,k)} |g| \le |x(k)| \left(\int_{[k-1,k)} |g|^2 \right)^{1/2} \le |x(k)| = \int_{[k-1,k)} x(k)\phi(z) = \int_m^\ell \phi(x)\phi(z)$$

where $z = (z(i)) \in K_0$ is defined by $z(i) = 0$ for all $i \ne k$, $z(k) = |x(k)|/x(k)$ if $x \ne 0$, $z(k) = 0$ if $x = 0$. Assume that (5) is valid for some $n \in \mathbb{N} \cup \{0\}$. Let $g \in L_{n+1}$ and $m < \ell$ in $\mathbb{N}$. There exist $g_1, g_2, \ldots, g_N \in L_n$ and $(n_i)_{i=1}^{N-1} \subset \mathbb{N}$ such that $\text{supp}(g_1) < n_1 \le \text{supp}(g_2) < n_2 \le \cdots < n_{N-1} \le \text{supp}(g_N)$ and $g = \sum_{i=1}^{N} y(i)g_i$. Thus if we set $n_0 = 0$ and we adopt the

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convention that $[s, t) = \emptyset$ whenever $s \geq t$, then

$$
\begin{aligned}
\int_{[m,\ell)} \phi(x)g &= \sum_{i=1}^{N} y(i) \int_{[m,\ell)} \phi(x)g_i \\
&= \sum_{i=1}^{N} y(i) \int_{[m \vee n_{i-1}, \ell \wedge n_i)} \phi(x)g_i \\
&\leq \sum_{i=1}^{N} y(i) \sup_{z_i \in K_n} \int_{[m \vee n_{i-1}, \ell \wedge n_i)} \phi(x)\phi(z_i),
\end{aligned}
$$

where the last inequality follows by the inductive hypothesis. Thus we continue our estimates as follows:

$$
\begin{aligned}
\int_{[m,\ell)} \phi(x)g &= \sum_{i=1}^{N} y(i) \sup_{\substack{z_i \in K_n \\ \text{supp}(z_i) \in [n_{i-1}+1, n_i]}} \int_{[m,\ell)} \phi(x)\phi(z_i) \\
&\leq \sup \left\{ \int_{[m,\ell)} \phi(x)\phi\left(\sum_{i=1}^{N} y(i)z_i \right) : z_i \in K_n \text{ for all } i, \text{ and} \right. \\
&\qquad \left. \text{supp}(z_1) \leq \tilde{n}_1 < \text{supp}(z_2) \leq \tilde{n}_2 < \cdots \leq \text{supp}(z_N) \text{ for some } (\tilde{n}_i) \subseteq \mathbb{N} \right\} \\
&= \sup_{z \in K_{n+1}} \int_{[m,\ell)} \phi(x)\phi(z)
\end{aligned}
$$

which finishes the inductive proof of (5).

Now, by combining (3) and (5) we obtain that $\phi \upharpoonright_{c_{00}} : (c_{00}, \|\cdot\|_y) \to (\mathcal{B}, |\cdot|_y)$ is an isometric embedding. Indeed, for every $x = (x(i)) \in c_{00}$ we have that

$$
\begin{aligned}
\|x\|_y &= \sup_{z=(z(i)) \in K} \sum_{i=1}^{\infty} x(i)z(i) \\
&= \sup_{z=(z(i)) \in K} \int_0^{\infty} \phi(x)\phi(z) \text{ (by (3))} \\
&= \sup_{g \in L} \int_0^{\infty} \phi(x)g \text{ (by (5))} \\
&= |\phi(x)|_y.
\end{aligned}
$$

Step 2: We define a norm $||| \cdot |||_y$ on $L_2[0, \infty)$ in terms of its dual ball as follows. Let

$$
L_0' = \{f \in L_2[0, \infty) : \|f\|_2 \leq 1 \text{ and } \text{supp}(f) \subseteq [\alpha, \alpha+1) \text{ for some } \alpha \geq 0\}.
$$

If $n \in \mathbb{N} \cup \{0\}$ and L_n' has been defined then

$$
L_{n+1}' = L_n' \cup \{\sum_{i=1}^{\infty} y(i)f_i : f_i \in L_n' \text{ for all } i \text{ and } \text{supp}(f_1) < \text{supp}(f_2) < \cdots < \text{supp}(f_N)\}.
$$

Let $L' = \cup_{n=0}^{\infty} L'_n$ and for a function $f \in L_2[0,\infty)$ define

$$|||f|||_y = \sup_{g \in L'} \int_0^\infty fg.$$

Notice that L' is a subset of the unit ball of $(L_2[0,\infty), \|\cdot\|_2)$ thus $|||\cdot|||_y \leq \|\cdot\|_2$. Also notice that for every $g \in L'$ there exists a finite tree $\mathcal{T}$ and functions $(g_t)_{t \in \mathcal{T}}$ with

(6) $$g_\emptyset = g,$$

(7) $\quad g_t = \sum_{i=1}^N y(i) g_{t_i}$ with $\operatorname{supp}(g_{t_1}) < \operatorname{supp}(g_{t_2}) < \cdots < \operatorname{supp}(g_{t_N})$
$\quad\quad$ for every $t \in \mathcal{T} \setminus \max(\mathcal{T})$,

(where $t^+ = \{t_1, \ldots, t_N\}$), and

(8) $\quad \|g_t\|_2 \leq 1$ and there exists $\alpha_t \geq 0$ such that $\operatorname{supp}(g_t) \subseteq [\alpha_t, \alpha_t + 1)$
$\quad\quad$ for every $t \in \max(\mathcal{T})$.

We now show that L' is a dense subset of $\{f \in \mathcal{B} : \|f\|_2 \leq 1\}$. Since $\{f \in \mathcal{B} : \|f\|_2 \leq 1\}$ is a dense subset of the unit ball of $(L_2[0,\infty), \|\cdot\|_2)$, we obtain that $|||\cdot|||_y$ is equal to $\|\cdot\|_2$. Now let $f \in \mathcal{B}$ with $\|f\|_2 \leq 1$. We define a (perhaps *infinite*) tree $\mathcal{T}$, a $\mathcal{T}$-decomposition $(E_t)_{t \in \mathcal{T}}$ of $[\inf(\operatorname{supp}(f)), \sup(\operatorname{supp}(f)))$, and for every $t \in \mathcal{T}$ we define $f_t \in L_2[0,\infty)$ with $\operatorname{supp}(f_t) \subseteq E_t$ as follows. Let $f_\emptyset = f$ and $E_\emptyset = [\inf(\operatorname{supp}(f)), \sup(\operatorname{supp}(f)))$. Assume that f_t has been defined for some $t \in \mathcal{T}$, and $\|f_t\|_2 = \|f\|_2$. Then t is a maximal node of $\mathcal{T}$ if there exists $\alpha \geq 0$ such that $\operatorname{supp}(f_t) \subseteq [\alpha, \alpha + 1)$. Otherwise, let $\alpha_0 = \inf(E_t)$ and define $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N = \sup(E_t)$ (where $N = \#\operatorname{supp}(y)$) such that $\|f_t \chi_{[\alpha_{i-1}, \alpha_i)}\|_2 = y(i) \|f_t\|_2$ for all $i = 1, \ldots, N$. Notice that this task is feasible since $\sum_{i=1}^n y(i)^2 = 1$ and the function $\alpha \mapsto \|f_t \chi_{[\alpha_{i-1}, \alpha)}\|_2$ is continuous. Then define $E_{t_i} = [\alpha_{i-1}, \alpha_i)$ and $f_{t_i} = \|f_t\|_2 f_t \chi_{[\alpha_{i-1}, \alpha_i)} / \|f_t \chi_{[\alpha_{i-1}, \alpha_i)}\|_2$ for $i = 1, \ldots, N$ and $t^+ = \{t_1, \ldots, t_N\}$. Notice that $\|f_{t_i}\|_2 = \|f_t\|_2 = \|f\|_2$ for all i, and

$$\sum_{i=1}^N y(i) f_{t_i} = \sum_{i=1}^N y(i) \|f_t\|_2 \frac{f_t \chi_{[\alpha_{i-1}, \alpha_i)}}{\|f_t \chi_{[\alpha_{i-1}, \alpha_i)}\|_2} = \sum_{i=1}^N f_t \chi_{[\alpha_{i-1}, \alpha_i)} = f_t.$$

It is easy to see by induction on $n = 0, 1, 2, \ldots$ that for every $t \in \mathcal{T}$ which has n predecessors, $\|f \chi_{E_t}\|_2 = \|f\|_2 y(j_1) y(j_2) \cdots y(j_n)$ for some $j_1, j_2, \ldots, j_n \in \{1, \ldots, N\}$ (which depend on t). Thus

(9) $\quad \|f \chi_{E_t}\| \to 0$ as n (the number of predecessors of t) tends to infinity.

Notice also that for every two incomparable nodes $t, s \in \mathcal{T}$ we have that the intervals E_t and E_s are disjoint. Thus there exist at most $\sup(E_\emptyset) - \inf(E_\emptyset)$ many t's in $\mathcal{T}$ such that the length of E_t is larger than 1. Thus by (9) we obtain that

$$g := f - \sum_{\substack{(\text{length of } E_t) > 1}} f \chi_{E_t} + \sum_{\substack{(\text{length of } E_t) > 1}} \|f \chi_{E_t}\|_2^2 \chi_{[\inf(E_t), \inf(E_t) + 1)}$$

is a good approximant of f if every $t \in \mathcal{T}$ for which (length of E_t) > 1 has sufficiently large number of predecessors. Moreover, working as above we obtain a *finite* tree $\mathcal{T}$ and functions $(g_t)_{t \in \mathcal{T}}$ which satisfy (6), (7) and (8). Thus $g \in L'$ which implies that L' is dense in $\{f \in \mathcal{B} : \|f\|_2 \leq 1\}$.

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Step 3: We now prove that $|||\cdot|||_y$ is equivalent to $|\cdot|_y$. Obviously we have that $|\cdot|_y \leq |||\cdot|||_y$. Let $a := \min_{1 \leq i \leq N} y(i)$ and $b := \max_{1 \leq i \leq N} y(i)$. Let $M \in \mathbb{N}$ such that $b^M \leq a$. We will show that every function $g \in L'$, the norming set of $|||\cdot|||_y$, can be written as a sum of at most $(M+4)N$ many functions from L which will finish the proof. Fix $g \in L'$ and consider an N-tree $\mathcal{T}$ and functions $(g_t)_{t \in \mathcal{T}} \subset L_2[0, \infty)$ which satisfy (6), (7) and (8). In particular, $(g_t)_{t \in \mathcal{T}}$ is a $\mathcal{T}$-decomposition of g. By (8), for every $t \in \max(\mathcal{T})$ there exists $\alpha_t \geq 0$ such that $\mathrm{supp}\,(g_t) \subseteq [\alpha_t, \alpha_t + 1)$. Notice that if $\alpha_t \in \mathbb{N} \cup \{0\}$ for every $t \in \max(\mathcal{T})$ and moreover for every $s, t \in \max(\mathcal{T})$ with $s \neq t$ we have that $\alpha_t \neq \alpha_s$ then $g \in L$. For every $i = 1, 2, \ldots, N$ let $\mathcal{T}_i \subseteq \mathcal{T}$ to be the smallest tree which contains all maximal nodes of $\mathcal{T}$ of the form $(n_1, \ldots, n_k, i)$. Thus all maximal nodes of $\mathcal{T}_i$ have the form $(n_1, \ldots, n_k, i)$ for some $k \in \mathbb{N}$ and $n_j \in \{1, \ldots, N\}$ for $1 \leq j \leq k$. For every $t \in \mathcal{T}$ let E_t to be the smallest interval (closed from the left, open from the right) containing the support of g_t. For $t \in \mathcal{T} \setminus \max(\mathcal{T})$ we have that the length of E_t is larger than 1. For every $i = 1, \ldots, N$ let $A_i, B_i \subseteq \max(\mathcal{T}_i)$ with $A_i \cup B_i = \max(\mathcal{T}_i)$ be defined by $A_i = \{t \in \max(\mathcal{T}_i) : E_t \cap \mathbb{N} = \emptyset\}$ and $B_i = \max(\mathcal{T}_i) \setminus A_i$. For every $t \in B_i$ let $n(t) \in \mathbb{N} \cap E_t$, (such $n(t)$ is unique since the length of E_t is at most equal to 1). We have that

$$\mathrm{supp}\,(g) \subseteq \bigcup_{i=1}^{N} \left\{ \bigcup\{E_t : t \in A_i\} \cup \bigcup\{E_t \cap [0, n(t)) : t \in B_i\} \cup \bigcup\{E_t \cap [n(t), \infty) : t \in B_i\} \right\}.$$

For $i \in \{1, \ldots, N\}$ let F_i be the restriction of g on $\cup\{E_t : t \in A_i\}$, G_i be the restriction of g on $\cup\{E_t \cap [0, n(t)) : t \in B_i\}$ and H_i be the restriction of g on $\cup\{E_t \cap [n(t), \infty) : t \in B_i\}$. Notice that for any fixed $i \in \{1, \ldots, N\}$, any two intervals of the form $E_t \cap [0, n(t))$, where $t \in B_i$, are separated with an integer. The same is true for any two intervals of the form $E_t \cap [n(t), \infty)$, where $t \in B_i$. Thus for $i \in \{1, \ldots, N\}$, each of the functions G_i and H_i has a $\mathcal{T}_i$-tree decomposition (naturally inherited from the $\mathcal{T}$-tree decomposition of g), such that if $(G_{i,t})_{t \in \mathcal{T}_i}$ and $(H_{i,t})_{t \in \mathcal{T}_i}$ are the tree decompositions of G_i and H_i respectively, then the following is satisfied. For any $s, t \in \max(\mathcal{T}_i)$ with $s \neq t$ and $G_{i,t}, G_{i,s} \neq 0$, we have that the supports of $G_{i,s}$ and $G_{i,t}$ are separated by an integer. Similarly, for any $s, t \in \max(\mathcal{T}_i)$ with $s \neq t$ and $H_{i,t}, H_{i,s} \neq 0$, we have that the supports of $H_{i,s}$ and $H_{i,t}$ are separated by an integer. Thus for $i \in \{1, \ldots, N\}$, G_i, H_i belong to L, the norming set for $|\cdot|_y$.

Unfortunately we cannot say the same about F_i! Indeed for $i \in \{1, \ldots N\}$, let $(F_{i,t})_{t \in \max \mathcal{T}_i}$ be the $\mathcal{T}_i$-tree decomposition of F_i which is naturally inherited by the $\mathcal{T}$-tree decomposition of g. In general for $s, t \in \max(\mathcal{T}_i)$ with $s \neq t$, the supports of $F_{i,s}$ and $F_{i,t}$ may not be separated by an integer. Indeed, let $s, t \in \max(\mathcal{T}_i)$ be such that $s \neq t$, $F_{i,t} \neq 0$, $F_{i,s} \neq 0$ and the sets $(s^-)^+$ and $(t^-)^+$ contain only maximal elements of $\mathcal{T}_i$. These conditions guarantee that E_{t^-} and E_{s^-} are disjoint intervals, each of length larger than 1. Also $\emptyset \neq E_t \subseteq E_{t^-}$ and $\emptyset \neq E_s \subseteq E_{s^-}$. It is easy to see that these conditions do not imply that E_t and E_s are separated by an integer. On the other hand there are no three different nodes $s, t, w \in \max(\mathcal{T}_i)$ so that $F_{i,s}, F_{i,t}, F_{i,w} \neq 0$, the sets $(s^-)^+, (t^-)^+, (w^-)^+$ contain only maximal nodes of $\mathcal{T}_i$ and E_t, E_s, E_w are contained in the same integer interval (i.e. an interval of the form $[m, m+1)$ for some $m \in \mathbb{N} \cup \{0\}$). Thus if from each integer interval we extract the "most left" and the "most right" interval E_t for $t \in \max(\mathcal{T}_i)$, then there is no other node s in $\max(\mathcal{T}_i)$ such that $E_s \neq \emptyset$, $(s^-)^+$ has only maximal nodes and $E_s \subseteq [m, m+1)$. This is the task of the next two paragraphs. For the rest of the proof, fix $i \in \{1, \ldots, N\}$ and let $(F_{i,t})_{t \in \mathcal{T}_i}$ be the natural $\mathcal{T}_i$-tree decomposition of F_i which is inherited by the $\mathcal{T}$-decomposition of g.

Fix $t \in A_i$. There exists a unique integer interval $[m, m+1)$ with $E_t \subseteq [m, m+1)$. The node t will be called *left* (perhaps the term i-left would be more precise, but since i is fixed we do not want to make the terminology too long) if there is no $s \in A_i$ such that $m \leq E_s < E_t$ and E_s is non-empty. Let L_i be the restriction of F_i on $\cup \{E_t : t \text{ is left } \}$. The function L_i has a $\mathcal{T}_i$-tree decomposition $(L_{i,t})_{t \in \mathcal{T}_i}$ which is naturally inherited by the $\mathcal{T}_i$-tree decomposition of F_i. For any two left nodes t_1, t_2 with $L_{i,t_1}, L_{i,t_2} \neq 0$, we have that the supports of L_{i,t_1} and L_{i,t_2} are separated by an integer, thus L_i belongs to L, the norming set for $|\cdot|_y$.

Fix $t \in A_i$ and let $[m, m+1)$ be the unique integer interval with $E_t \subseteq [m, m+1)$. The node t will be called *right* if t is not left and there is no $s \in A_i$ such that $E_t < E_s < m+1$ and E_s is non-empty. Let R_i be the restriction of F_i on $\cup \{E_t : t \text{ is right } \}$. Arguing similarly as we did for L_i, we have that $R_i \in L$.

Let $\tilde{F}_i := F_i - (L_i + R_i)$ and $(\tilde{F}_{i,t})_{t \in \mathcal{T}_i}$ be the $\mathcal{T}_i$-tree decomposition of $\tilde{F}_i$ which is naturally inherited by the $\mathcal{T}_i$-tree decomposition of F_i. For $s, t \in \max(\mathcal{T}_i)$ with $s \neq t$ and $\tilde{F}_{i,s}, \tilde{F}_{i,t} \neq 0$, the supports of $F_{i,s}$ and $F_{i,t}$ may not be separated by an integer! For instance consider the following scenario. Consider two maximal nodes s, t of $\mathcal{T}_i$ with $s^- \prec t^-$. Since $\operatorname{supp}(\tilde{F}_{i,t^-}) \subset \operatorname{supp}(\tilde{F}_{i,s^-})$, and $\operatorname{supp}(\tilde{F}_{i,t^-})$ is not contained in any integer interval we have that $\operatorname{supp}(\tilde{F}_{i,s^-})$ is not contained in any integer interval either. Obviously the inclusions $\operatorname{supp}(\tilde{F}_{i,s}) \subseteq \operatorname{supp}(\tilde{F}_{i,s^-})$ and $\operatorname{supp}(\tilde{F}_{i,t}) \subseteq \operatorname{supp}(\tilde{F}_{i,t^-})$ do not imply that the sets $\operatorname{supp}(\tilde{F}_{i,s})$ and $\operatorname{supp}(\tilde{F}_{i,t})$ are separated by an integer! The special nodes s, t that we just considered have to be treated in a special way, and that is what we do in what follows.

A sequence $(t_1, t_2, \ldots, t_k) \subset \mathcal{T}_i$, $(k \geq 2)$, is called *i-special* if the following are satisfied:

(a) $t_j \in \max(\mathcal{T}_i)$ for $1 \leq j \leq k$.
(b) $t_1^- \prec t_2^- \prec \cdots \prec t_k^-$.
(c) The sets $\operatorname{supp}(\tilde{F}_{i,t_j})$ for $j \in \{1, \ldots, k\}$ are all contained in the same integer interval.
(d) None of the t_j's is left or right.
(e) The sequence $(t_1, t_2, \ldots, t_k)$ is maximal with the properties (a), (b), (c) and (d) (i.e. it is not properly contained in any sequence which satisfies (a), (b), (c) and (d)).

Properties (c), (d) and (e) ensure that any two i-special sequences are disjoint.

Define $\mathcal{T}_{i,1}$ to be the smallest subtree of $\mathcal{T}_i$ satisfying the following two conditions:

(i) If $t \in \max(\mathcal{T}_i)$ and t does not belong in any i-special sequence then $t \in \mathcal{T}_{i,1}$.
(ii) For any i special sequence $(t_1, \ldots, t_k)$ we have that $t_1 \in \mathcal{T}_{i,1}$.

Recall that in the beginning of Step 3 we defined a, b and M by: $a := \min_{1 \leq \ell \leq N} y(\ell)$, $b := \max_{1 \leq \ell \leq N} y(\ell)$ and $M \in \mathbb{N}$ such that $b^M \leq a$. For $j \in \{2, \ldots, M-1\}$ let $\mathcal{T}_{i,j}$ be the smallest subtree of $\mathcal{T}_i$ whose maximal nodes are the elements t_j of any i-special sequence $(t_1, \ldots, t_k)$ with $k \geq j$. By the definition of $\mathcal{T}_{i,j}$ for $1 \leq j \leq M-1$ we have that if $(t_1, \ldots, t_k)$ is an i-special sequence and $1 \leq j \leq k$ then $t_\ell \notin \mathcal{T}_{i,j}$ for $\ell \in \{1, \ldots k\} \setminus \{j\}$. For every $j \in \{1, \ldots, M-1\}$ let $\tilde{F}_{i,j}$ be the restriction of $\tilde{F}_i$ on $\cup \{E_t : t \in A_i \cap \mathcal{T}_{i,j}\}$. Hence $\tilde{F}_{i,j}$ has a $\mathcal{T}_{i,j}$-tree decomposition $(\tilde{F}_{i,j,t})_{t \in \mathcal{T}_{i,j}}$ which is naturally inherited by the $\mathcal{T}_i$-tree decomposition of $\tilde{F}_i$. Moreover, since i-special sequences are maximal, (see (e) in the definition of i-special sequences), we have that no two maximal nodes of $\mathcal{T}_{i,j}$ belong in the same i-special sequence. Hence for every two maximal nodes s, t of $\mathcal{T}_{i,j}$ we have that the supports of $\tilde{F}_{i,j,s}$ and $\tilde{F}_{i,j,t}$ are separated by an integer. Thus $\tilde{F}_{i,j}$ belongs to the norming set L of $|\cdot|_y$ for every $j \in \{1, \ldots, M-1\}$.

Notice that $\cup_{j=1}^{M-1}\mathrm{supp}\,(\tilde{F}_{i,j})$ may be a strict subset of $\mathrm{supp}\,(\tilde{F}_i)$ and thus $\tilde{F}_i$ may not be equal to $\sum_{j=1}^{M-1}\tilde{F}_{i,j}$. This will be the case if there are i-special sequences that have length $k \geq M$. Now we take care of such long i-special sequences. Fix an i-special sequence $(t_1,\ldots,t_k)$ with $k \geq M$. Let $[m,m+1)$ be the unique integer interval with $\mathrm{supp}\,(\tilde{F}_{i,t_j}) \subseteq [m,m+1)$ for all $j \in \{1,\ldots,k\}$. Let $\tilde{F}_{i,t_1^-}\restriction_{[m,m+1)}$ and $\tilde{F}_{i,t_M^-}\restriction_{[m,m+1)}$ be the restrictions of $\tilde{F}_{i,t_1^-}$ and $\tilde{F}_{i,t_M^-}$ respectively on $[m,m+1)$. Thus we can write

$$\tilde{F}_{i,t_1^-}\restriction_{[m,m+1)}= \left(\prod y(n_\alpha)^{m_\alpha}\right)\tilde{F}_{i,t_M^-}\restriction_{[m,m+1)}$$

where $n_\alpha, m_\alpha \in \{1,\ldots,N\}$ and the product $\prod y(n_\alpha)^{m_\alpha}$ has at least M-terms counting multiplicities. Thus $\prod y(n_\alpha)^{m_\alpha} \leq b^M \leq a \leq y(i)$. Hence $\|\tilde{F}_{i,t_1^-}\restriction_{[m,m+1)}\|_2 \leq y(i)\|\tilde{F}_{i,t_M^-}\restriction_{[m,m+1)}\|_2$ and $\|\tilde{F}_{i,t_M^-}\restriction_{[m,m+1)}\|_2 \leq \|\tilde{F}_{i,t_M^-}\|_2 \leq 1$ by (8). Let $\mathcal{T}_{i,M}$ be the subtree of $\mathcal{T}_{i,1}$ whose maximal nodes are the nodes t_1 for any i-maximal sequence $(t_1,\ldots,t_k)$ with $k \geq M$. Let

$$S_i := \cup\{E_t : t \in A_i \text{ and there exists an } i\text{-special sequence } (t_1,\ldots,t_k) \text{ with } k \geq N$$

$$\text{and } M \leq j \leq k \text{ such that } t = t_j\}.$$

Thus the restriction $\tilde{F}_{i,M} := \tilde{F}_i \restriction_{S_i}$ of the function $\tilde{F}_i$ on S_i has a $\mathcal{T}_{i,M}$-tree decomposition $(\tilde{F}_{i,M,t})_{t\in\mathcal{T}_{i,M}}$, where for any i-special sequence $(t_1,\ldots,t_k)$ with $k \geq M$ we set $\tilde{F}_{i,M,t_1} := \tilde{F}_{i,t_M^-}\restriction_{[m,m+1)}$, (where $[m,m+1)$ is the unique integer interval which contains the sets $\mathrm{supp}\,(\tilde{F}_{i,t_j})$ for $j \in \{1,\ldots,k\}$) and $\tilde{F}_{i,M,t_1^-} := \tilde{F}_{i,t_1^-}\restriction_{[m,m+1)}$. Hence $\tilde{F}_{i,M}$ belongs to L the norming set of $|\cdot|_y$.

We have decomposed the arbitrary element g of the norming set L' of $|||\cdot|||_y$ into the sum of the functions G_i, H_i, L_i, R_i, $\tilde{F}_{i,j}$ (where $i \in \{1,\ldots,N\}$ and $j \in \{1,\ldots,M\}$), a total of $(M+4)N$ many functions of the norming set L of $|\cdot|_y$. This implies that $|||\cdot|||_y \leq (M+4)N|\cdot|_y$ and finishes the proof. $\square$

References

[1] Odell, E. ; Schlumprecht, Th. *The distortion problem.* Acta Math. **173** (1994), no. 2, 259–281.
[2] Odell, E. ; Schlumprecht, Th. *Distortion and stabilized structure in Banach spaces; new geometric phenomena for Banach and Hilbert spaces.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 955–965, Birkhäuser, Basel, 1995.
[3] Bellenot, S.F. *Tsirelson superspaces and ℓ_p.* J. Funct. Anal. **69** (1986), no. 2, 207–228.
[4] Bernués, J.; Deliyanni, I. *Families of finite subsets of $\mathbb{N}$ of low complexity and Tsirelson type space,* Math. Nachr. 222 (2001), 15–29.
[5] Gowers, W.T. *Ramsey methods in Banach spaces.* Handbook of the geometry of Banach spaces, Vol. **2**, 1071–1097, North-Holland, Amsterdam, 2003.
[6] Pelczar, A.M. *Stabilization of Tsirelson-type norms on ℓ_p spaces,* preprint, avalaible at http://www.arxiv.org/pdf/math.FA/0508352

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