ELTON NEAR UNCONDITIONALITY OF ARRAYS

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Abstract. There are many notions of partial unconditionality defined for a weakly null basic sequence in a Banach space. In 2008 the idea of Schreier unconditionality was extended to a structure in Banach spaces called arrays. Here we extend the idea of Elton near unconditionality to arrays.

0. Introduction

The mission of finding unconditionality in every Banach space ended in 1993 with the Banach space of Gowers and Maurey [6] in which there is no unconditional basic sequence. However the search for unconditionality was not in vain. There have been several partial results. And work in finding these partial unconditionalities continues in [2], [3] and [7]. Generally, these partial unconditionalities are found in every weakly null basic sequence. Two of the first such partial unconditionalities are: Elton δ-near unconditionality [4] (which can be found in [10]) and Schreier unconditionality (see [8], [11] and [9]). In 2008 in [1] Schreier unconditionality was extended to arrays in Banach spaces. Herein, we will extend δ-near unconditionality to arrays in Banach spaces.

First we define the structures we will need using the same definitions found in [1]. Let $I = \{(i,j) \in \mathbb{N}^2 : i \leq j\}$. Define an order on I (reverse lexicographical order) as

$$(i_1, j_1) <_{r\ell} (i_2, j_2) \text{ if and only if } \begin{cases} j_1 < j_2, \text{ or} \\ j_1 = j_2 \text{ and } i_1 < i_2. \end{cases}$$

We say an **array** is a collection of vectors $(x_{i,j})_{(i,j)\in I}$ in a Banach space so that for each $i_0 \in \mathbb{N}$ we have $(x_{i_0,j})_{j\geq i_0 : j\in\mathbb{N}}$ is a seminormalized weakly null sequence. A **subarray** of $(x_{i,j})_{(i,j)\in I}$ is an array $(y_{i,\ell})_{(i,\ell)\in I}$ so that for each $i_0 \in \mathbb{N}$ there is some increasing sequence $(n_\ell^{i_0})_{\ell=1}^\infty$ in $\mathbb{N}$ so that $y_{i_0,\ell} = x_{i_0, n_\ell^{i_0}}$ for all $\ell \in \mathbb{N}$. A **regular array** is an array when ordered with $<_{r\ell}$ is a basic sequence.

1. Theorem

The original theorem by Elton for sequences is as follows:

Theorem 1.1 (J. Elton). *Let (x_j) be a normalized weakly null basic sequence in a Banach space X. Then there exists a subsequence (y_j) of (x_j) such that for any $\delta > 0$ there is $C = C(\delta)$ so that for any $(a_n) \in c_{00}$ with $|a_n| \leq 1$ and for all $F \subseteq \{\operatorname{supp}(a_j) : |a_j| \geq \delta\}$*

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$$\|\sum_{i\in F} a_j y_j\| \leq C(\delta)\|\sum a_j y_j\|.$$

Moreover, $C = C(\delta)$ is independent of the sequence.

One of the powers of the array (as seen in [1]) is preserving the properties of each row (ie $(x_{i,j})_{j\geq i_0: j\in\mathbb{N}}$). We extend Theorem 1.1 to arrays and preserve the rows. That is, Elton type projections on rows are bounded.

Theorem 1.2. *Let $(x_{i,j})_{(i,j)\in I}$ be a normalized regular array in a Banach space X. Then there exists a subarray $(y_{i,j})_{(i,j)\in I}$ of $(x_{i,j})_{(i,j)\in I}$ such that for any $\delta > 0$ there is $C = C(\delta)$ so that for any $(a_{n,m}) \in c_{00}(I)$ with $|a_{n,m}| \leq 1$ and for any $k_0 \in \mathbb{N}$ and $F \subseteq \{(k_0, j) \in \operatorname{supp}(a_{i,j}) : |a_{i,j}| \geq \delta\}$*

$$\|\sum_{(k_0,i)\in F} a_{k_0,j} y_{k_0,j}\| \leq C(\delta)\|\sum a_{i,j} y_{i,j}\|.$$

Moreover, $C = C(\delta)$ is independent of the sequence.

2. proof

The main ingredient in the selection of the array is Ramsey theory; and the particular flavor of Ramsey theory we will be using is Theorem 2.1 by Galvin and Prikry [5]. For M, an infinite subset of $\mathbb{N}$, we say $[M] = \{(n_i)_{i=1}^{\infty} \subseteq M : n_1 < n_2 < n_3 < \cdots\}$. And we say $[M]^{\omega}$

Theorem 2.1. *Let $[\mathbb{N}]^{\omega} = P_0 \cup P_1 \cup \cdots \cup P_{k-1}$ where each P_i is Borel. Then there is an infinite $H \subset \mathbb{N}$ so that for some $i \in \{0, 1, \ldots, k-1\}$ we have $[P_i]^{\omega} \subset P_i$.*

Now a few general remarks about arrays (can be found in [1]).

Remark 2.2. *If $(x_{i,j})_{(i,j)\in I}$ is a regular array and $(y_{i,j})_{(i,j)\in I}$ is a subarray of $(x_{i,j})_{(i,j)\in I}$ then $(y_{i,j})_{(i,j)\in I}$ is also regular.*

Remark 2.3. *Let $(x_i)_{i=1}^{N}$ be a finite basic sequence in some infinite dimensional Banach space X having basis constant C. Let (y_i) be a seminormalized weakly null sequence X and $\varepsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $(x_1, x_2, \ldots, x_N, y_n)$ is a basic sequence with constant $C(1+\varepsilon)$.*

By repeated application of Remark 2.3 we obtain the following.

Remark 2.4. *Let X be a Banach space and for every $i \in \mathbb{N}$ let $(x_{i,j})_{j=i}^{\infty}$ be a seminormalized weakly null sequence in X. Then there exists a subarray $(y_{i,j})_{(i,j)\in I}$ of $(x_{i,j})_{(i,j)\in I}$ which is regular. Moreover, the basis constant of $(y_{i,j})_{(i,j)\in I}$ can be chosen to be arbitrarily close to 1.*

We begin with Lemma 2.5 which roughly says that given any regular array and some i_0 for any functional f there is another functional g which preserves the positive mass of f on some subset of I and is very small at some point (i_0, j_0) in the array. For $f \in X^*$ we say $f^+(x) = f(x)$ if $f(x) > 0$ and $f^+(x) = 0$ otherwise.

Lemma 2.5. *Let $(x_{i,j})_{(i,j)\in I}$ be a regular array in a Banach space X, $\mathcal{F} \subseteq 2Ba(X^*)$, $\delta > 0$, $i_0, k_0 \in \mathbb{N}$ and $K > 0$. Then there exists a subarray $(y_{i,j})_{(i,j)\in I}$ of $(x_{i,j})_{(i,j)\in I}$ such that for any $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{k_0, k_0+1, k_0+2, \ldots\}$, and any $j_0 \in \mathbb{N}$ where $(i_0, j_0) <_{r\ell} (k_0, b_1)$*

we have the following property:

If there exists $f \in \mathcal{F}$ *so that* $\sum_{j=1}^{p} f^+(y_{(k_0,b_j)}) \geq K$

then there exists $g \in \mathcal{F}$ *so that* $\sum_{j=1}^{p} g^+(y_{(k_0,b_j)}) \geq K$ *and* $|g(y_{(i_0,j_0)})| < \delta$.

Additionally if $(z_{i,j})_{(i,j)\in I}$ *is a subarray of* $(y_{i,j})_{(i,j)\in I}$ *the above property still holds.*

Proof. Let

$$S_k = \left\{ M = (m_i)_{i=1}^{\infty} \in [\mathbb{N}] : \text{if there is } f \in \mathcal{F} \text{ with } \sum_{i=2}^{k} f^+(x_{k_0,m_i}) \geq K \right.$$
$$\left. \text{then there is } g \in \mathcal{F} \text{ with } \sum_{i=2}^{k} g(x_{k_0,m_i}) \geq K \text{ and } |g(x_{i_0,m_1})| < \delta \right\}$$

Let $S = \cap_{k=1}^{\infty} S_k$. Notice each S_k is pointwise closed. Thus S is Borel and by Theorem 2.1 there is $M \in [\mathbb{N}]$ so that either $[M] \subseteq S$ or $[M] \subseteq [\mathbb{N}] \setminus S$.

Assume (toward contradiction) that there is an $M = (m_i) \in [\mathbb{N}]$ so that $[M] \subseteq [\mathbb{N}] \setminus S$. Let n be arbitrary and for each j with $1 \leq j \leq n$ define $L_j = \{m_j, m_{n+1}, m_{n+2}, \ldots\}$. So $L_j \notin S$. Thus for each j with $1 \leq j \leq n$ there is $f_j \in \mathcal{F}$ and $\ell_j \in \mathbb{N}$ so that

$$\sum_{i=2}^{\ell_j} f_j^+(x_{k_0,m_i}) \geq K$$

and $|f_j(x_{i_0,m_j})| \geq \delta$. Let $\ell_{j_0} = \min\{\ell_j\}$. For each j with $1 \leq j \leq n$ we have $L_j \notin S$ and $\sum_{i=2}^{\ell_{j_0}} f_{j_0}^+(x_{k_0,m_i}) \geq K$ thus $|f_{j_0}(x_{i_0,m_j})| \geq \delta$. Since n is arbitrary we have that for each n there is a $f_n \in 2Ba(x^*)$ with $|f_n(x_{i_0,m_j})| \geq \delta$ for all j with $1 \leq j \leq n$ contradicting $(x_{i_o,j})_{j=i_0}^{\infty}$ is weakly null. Therefore there is some $(m_i) \in [\mathbb{N}]$ so that $[(m_i)] \in S$. Define

$$y_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) <_{r\ell} (i_0,j_0), \\ x_{i_0,m_1} & \text{if } (i,j) = (i_0,j_0), \\ x_{i,m_{j-j_0+1}} & \text{if } (i,j) >_{r\ell} (i_0,j_0). \end{cases}$$

□

We continue by extending Lemma 2.5. In Lemma 2.6 we generate another regular subarray with the property that for any functional f we can find another functional g preserving the positive mass on that same subset of I. And additionally, we gain freedom as to the array vector that will be small on g.

Lemma 2.6. *Let* $(x_{i,j})_{(i,j)\in I}$ *be a regular array in a Banach space* X, $K < \infty$, $\mathcal{F} \subseteq 2Ba(X^*)$ *and* $\delta > 0$. *Then there exists a subarray* $(y_{i,j})_{(i,j)\in I}$ *of* $(x_{i,j})_{(i,j)\in I}$ *such that for any* $k_0 \in \mathbb{N}$, *any* $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{k_0, k_0+1, k_0+2, \ldots\}$ *and any* $(i_0, j_0) \in I$ *with* $(1, k_0) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, b_1)$ *we have the following:*

If there exists $f \in \mathcal{F}$ *so that* $\sum_{j=1}^{p} f^+(y_{(k_0,b_j)}) \geq K$

then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g(y_{(k_0,b_j)}) \geq K$ and $|g(y_{(i_0,j_0)})| < \delta$.

Proof. We will define a subarray recursively on the column index (that has been the second index j).

For $j = 1$ apply Lemma 2.5 to $(x_{i,j})_{(i,j)\in I}$, $\mathcal{F}$, δ, K, $i_0 = 1$, and $k_0 = 1$ to obtain a subarray $(y^1_{i,j})_{(i,j)\in I}$ of $(x_{i,j})_{(i,j)\in I}$ with the property that for any $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{1, 2, 3, \ldots\}$, and for any $(1, j_0) <_{r\ell} (1, \min(B))$ we have the following property:

If there exists $f \in \mathcal{F}$ so that $\sum_{j=1}^{p} f^+(y^1_{(1,b_j)}) \geq K$

then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g^+(y^1_{(1,b_j)}) \geq K$ and $|g(y^1_{(1,j_0)})| < \delta$.

For $j = 2$ apply Lemma 2.5 to $(y^1_{i,j})_{(i,j)\in I}$, $\mathcal{F}$, δ, K, and then successively for each pair $i_0 = 1$, and $k_0 = 2$, $i_0 = 2$, and $k_0 = 1$, and $i_0 = 2$, and $k_0 = 2$ to obtain a subarray $(y^2_{i,j})_{(i,j)\in I}$ of $(y^1_{i,j})_{(i,j)\in I}$ with the property that for any $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{1, 2, 3, \ldots\}$, where $i_0, k_0 \in \{1, 2\}$ and $(1, 2) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, \min(B))$ we have the following property:

If there exists $f \in \mathcal{F}$ so that $\sum_{j=1}^{p} f^+(y^2_{(k_0,b_j)}) \geq K$

then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g^+(y^2_{(k_0,b_j)}) \geq K$ and $|g(y^2_{(i_0,j_0)})| < \delta$.

For $j = r$ (for some $r > 1$) apply Lemma 2.5 to $(y^{r-1}_{i,j})_{(i,j)\in I}$, $\mathcal{F}$, δ, K, and then successively for each pair $(i_0, k_0) \in \{(i, j_0) : 1 \leq i < j_0\} \cup \{(j_0, j) : 1 \leq j \leq j_0\}$ to obtain a subarray $(y^r_{i,j})_{(i,j)\in I}$ of $(y^{r-1}_{i,j})_{(i,j)\in I}$ with the property that for any $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{1, 2, 3, \ldots\}$, where $i_0, k_0 \in \{1, 2, \ldots, k_0\}$ and $(1, r) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, \min(B))$ we have the following property:

If there exists $f \in \mathcal{F}$ so that $\sum_{j=1}^{p} f^+(y^{r-1}_{(k_0,b_j)}) \geq K$

then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g^+(y^{r-1}_{(k_0,b_j)}) \geq K$ and $|g(y^{r-1}_{(i_0,j_0)})| < \delta$.

Define the subarray $y_{i,j} = y^j_{i,j}$ for all $(i, j) \in I$ of $(x_{i,j})_{(i,j)\in I}$. We have built our subarray and now we will show it satisfies the criterion above.

Let $(i_0, j_0) \in I$, $k_0 \in \mathbb{N}$ and $B = \{b_1, b_2, \ldots, b_p\} \subseteq \{k_0, k_1, k_2, \ldots\}$ so that $(1, k_0) \leq_{r\ell} (i_0, j_0) <_{r\ell} (k_0, \min(B))$. Since $(y_{i,j})_{(i,j)\in I}$ is a subarray of $(y^{j_0}_{i,j})_{(i,j)\in I}$ we have for $B' = \{b'_1, b'_2, \ldots, b'_p\}$ where $y_{k_0,b_1} = y^{j_0}_{k_0,b'_1}$ so that
if there exists $f \in \mathcal{F}$ so that $\sum_{j=1}^{p} f^+(y^{j_0}_{(k_0,b'_j)}) \geq K$
then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g^+(y^{j_0}_{(k_0,b'_j)}) \geq K$ and $|g(y^{j_0}_{(i_0,j_0)})| < \delta$.

Thus If there exists $f \in \mathcal{F}$ so that $\sum_{j=1}^{p} f^+(y_{(k_0,b_j)}) \geq K$
then there exists $g \in \mathcal{F}$ so that $\sum_{j=1}^{p} g^+(y_{(k_0,b_j)}) \geq K$ and $|g(y^{j_0}_{(i_0,j_0)})| < \delta$.

$\square$

Lemma 2.7. *Let $(x_{i,j})_{(i,j)\in I}$ be a regular array in a Banach space X, $\varepsilon > 0$ and $\delta \in (0,1)$. Then there exists a subarray $(y_{i,j})_{(i,j)\in I}$ of $(x_{i,j})_{(i,j)\in I}$ so that if $f \in Ba(X^*)$, $k_0 \in \mathbb{N}$, $(\hat{i},\hat{j}) \in I$ and $B \subseteq \{j \in N : j < \hat{j}$ and $f(y_{k_0,j}) > 0\}$ satisfies*

$$\sum_{j\in B} f(y_{k_0,j}) > \varepsilon$$

then there is a $g \in Ba(X^*)$ with

$$\sum_{j\in B} g^+(y_{k_0,j}) > (1-\delta)\sum_{j\in B} f(y_{k_0,j}) \ \text{ and } \sum_{(i,j)\in C} |g(y_{i,j})| < \delta$$

where $C = \{(i,j) \in I : (i,j) <_{r\ell} (\hat{i},\hat{j})\} \setminus \{(k_0,j) \in I : j \in B$ and $g(y_{k_0,j}) \leq 0\}$. Additionally all further subarrays of $(y_{i,j})_{(i,j)\in I}$ also have this property.

Proof. Let $\delta_n = 2^{-k_n}$ where (k_n) is a fast increasing sequence in $\mathbb{N}$ that satisfies the following:

- $\delta_1 \leq \frac{\delta}{10}$ and
- $\sum j\delta_j \leq \frac{\delta\varepsilon}{10}$.

Let $A_\ell = \big\{ -1 + k\delta_\ell : k \in \{0,1,2,\dots 2^{k_\ell+1}\}\big\}$. So each A_ℓ is an δ_ℓ-net for $[-1,1]$. We will construct $(y_{i,j})$ inductively.

For each $a \in A_1$ define

$$\mathcal{F}_a = \{f \in Ba(X^*) : f(x_{1,1}) \in a\}.$$

And for each $a \in A_1$ we apply Lemma 2.6 to $(x_{i,j})_{(i,j)\in I}$, $\delta = \delta_2$, $\mathcal{F} = \mathcal{F}_a$ and $K = k\delta_2$ for $k \in \{1,2,\dots,\delta_2^{-2}\}$ to obtain a subarray $(y^{1,1}_{i,j})_{(i,j)\in I}$ so that for any $k_0 \in \mathbb{N}$, any $B = \{b_1,b_2,\dots,b_p\} \subseteq \{k_0,k_0+1,k_0+2,\dots\}$ and any $(i_0,j_0) \in I$ with $(1,k_0) \leq_{r\ell} (i_0,j_0) <_{r\ell} (k_0,\min(F))$ we have the following: If there exists $f \in \mathcal{F}$ so that $\sum_{j\in B}^{p} f^+(y_{(k_0,b_j)}) \geq K$ then there exists $g \in \mathcal{F}$ so that $\sum_{j=2}^{p} g(y_{(k_0,b_j)}) \geq K$ and $|g(y_{(i_0,b_1)})| < \delta$.
Set $y_{1,1} = y^{1,1}_{1,1}$.

For each $\vec{a} \in A1 \times A_2$ define

$$\mathcal{F}_{\vec{a}} = \{f \in Ba(X^*) : f(y^{1,1}_{1,1}) \in a(1) \text{ and } f(y^{1,1}_{1,2}) \in a(2)\}.$$

And for each $\vec{a} \in A1 \times A_2$ we apply Lemma 2.6 to $(y^{1,2}_{i,j})_{(i,j)\in I}$, $\delta = \delta_2$, $\mathcal{F} = \mathcal{F}_a$ and $K = k\delta_2$ for $k \in \{1,2,\dots,\delta_2^{-2}\}$ to obtain a subarray $(y^{1,2}_{i,j})_{(i,j)\in I}$ so that for any $k_0 \in \mathbb{N}$, any $B = \{b_1,b_2,\dots,b_p\} \subseteq \{k_0,k_0+1,k_0+2,\dots\}$ and any $(i_0,j_0) \in I$ with $(1,k_0) \leq_{r\ell} (i_0,j_0) <_{r\ell} (k_0,\min(F))$ we have the following: If there exists $f \in \mathcal{F}$ so that $\sum_{j\in B}^{p} f^+(y^{1,2}_{(k_0,b_j)}) \geq K$ then there exists $g \in \mathcal{F}$ so that $\sum_{j=2}^{p} g(y^{1,2}_{(k_0,b_j)}) \geq K$ and $|g(y_{(i_0,b_1)})| < \delta$.
Set $y_{1,2} = y^{1,2}_{1,2}$.

Continue "walking" through the entire index set I in $<_{r\ell}$ order setting a vector in our $(y_{i,j})_{(i,j)\in I}$ after each step.

Now that we have finished generating the subarray we move to proving the conclusion. So let $k_0 \in \mathbb{N}$. Let $\bar{f} \in Ba(X^*)$ and $B \subseteq \{j \leq \hat{j} : \bar{f}(y_{k_0,j}) > 0\}$ so that $\sum_{j\in B} \bar{f}(y_{k_0,j}) > \varepsilon$. Let j^* be minimal so that

$$\sum_{j\in B} \bar{f}(y_{k_0,j}) \leq \frac{1}{\delta_{j^*}}.$$

So $\sum_{j\in B} \bar{f}(y_{k_0,j}) > \frac{1}{\delta_{j^*-1}}$ if $j^* > 1$. Note

$$\begin{aligned}\sum_{j\in B: j\geq j^*} \bar{f}(y_{k_0,j}) &\geq \sum_{j\in B} \bar{f}(y_{k_0,j}) - (j^*-1) = \sum_{j\in B} \bar{f}(y_{k_0,j}) - (j^*-1)\delta_{j*-1}\frac{1}{\delta_{j*-1}} \\ &\geq \sum_{j\in B} \bar{f}(y_{k_0,j}) - (j^*-1)\delta_{j*-1}\sum_{j\in B} \bar{f}(y_{k_0,j}) \\ &\geq (1-\delta/10)\sum_{j\in B} \bar{f}(y_{k_0,j}) \text{ since } (j^*-1)\delta_{j*-1} < \delta/10.\end{aligned}$$

Since $(y_{i,j})_{(i,j)\in I}$ is bimonotone there is $f \in Ba(X^*)$ so that

- $f(y_{i,j}) = 0$ for all $(i,j) <_{r\ell} (k_0, j*)$ and
- $f(y_{i,j}) = \bar{f}(y_{i,j})$ for all $(i,j) \geq_{r\ell} (k_0, j*)$.

So $\sum_{j\in B: j\geq j^*} \bar{f}(y_{k_0,j}) = \sum_{j\in B: j\geq j^*} f(y_{k_0,j})$. We will "walk" through the array to generate our functional g that satisfies the conclusion of the lemma.

STEP (k_0, j^*)**:** If $(k_0, j^*) \in \{(k_0, j) : j \in B\}$ then let

$$g_{k_0,j^*} = f \text{ and } B_{k_0,j^*} = \{j > j^* : j \in B\}.$$

Otherwise we have $(k_0, j^*) \notin \{(k_0, j) : j \in B\}$. Let k be maximal so that $\sum_{j\in B; j>j^*} f(y_{k_0,j}) > k\delta_{j^*}$ and $\mathcal{F}_{\vec{a}}$ where $\vec{a} = f(y_{1,1}) \times f(y_{1,2}) \times f(y_{2,2}) \times \cdots \times f(y_{k_0,j^*})$. By conclusion of Lemma 2.6 where $\mathcal{F} = \mathcal{F}_{\vec{a}}$ and $K = k\delta_{j^*}$, since $f \in \mathcal{F}_{\vec{a}}$ there exists $g_{k_0,j^*} \in \mathcal{F}_{\vec{a}}$ so that:

- $\sum_{j\in B; j>j^*} g^+_{k_0,j^*}(y_{k_0,j}) > k\delta_{j^*}$ and
- $|g_{(k_0,j^*)}(y_{k_0,j})| < \delta_{j^*}$.

Now we proceed to the inductive step case taking care to note that $j \in B_{i',j'}$ we know $f(y_{k_0,j}) > 0$ but we do not know if $g_{i',j'}(y_{k_0,j})$ is positive or negative.

STEP $(i', j') + 1$ **(given** (i', j')**):** If $(i', j') \in \{(k_0, j) : j \in B_{i',j'}\}$ and $g_{i',j'}(y_{k_0,j}) \geq 0$ then let

$$g_{(i',j')+1} = g_{(i',j')}.$$

Otherwise we have either $(i', j') \notin \{(k_0, j) : j \in B\}$ or $(i', j') \in \{(k_0, j) : j \in B\}$ but $g_{i',j'}(y_{k_0,j}) < 0$. Let k be maximal so that $\sum_{j\in B_{i',j'}} g_{i',j'}(y_{k_0,j}) > k\delta_{j'}$ and $\mathcal{F}_{\vec{a}}$ where $\vec{a} = f(y_{1,1}) \times f(y_{1,2}) \times f(y_{2,2}) \times \cdots \times f(y_{i',j'})$. By conclusion of Lemma 2.6 where $\mathcal{F} = \mathcal{F}_{\vec{a}}$ and $K = k\delta_{j'}$, since $f \in \mathcal{F}_{\vec{a}}$ there exists $g_{(i',j')+1} \in \mathcal{F}_{\vec{a}}$ so that:

- $\sum_{j\in B_{i',j'}} g^+_{(i',j')+1}(y_{k_0,j}) > k\delta_{j'}$ and
- $|g_{(i',j')}(y_{(i',j')+1})| < \delta_{j'}$.

Thus $\sum_{j\in B_{i',j'}} g^+_{(i',j')}(y_{k_0,j}) - \sum_{j\in B_{i',j'}} g^+_{(i',j')+1}(y_{k_0,j}) > k\delta_{j'} + \delta_{j'} > 0$.

And in either case set $B_{k_0,j^*} = \{j > j^* : j \in B \text{ and } (k_0, j) >_{r\ell} (i', j')\}$. Continue this process until $(i', j') = (\hat{i}, \hat{j})$ and set $g = g_{\hat{i},\hat{j}}$. Note

$$|g(y_{i,j}) - f(y_{i,j})| \leq \delta_j \text{ for } (i,j) <_{r\ell} (k_0, j*).$$

Define $P = \{j \in B_{k_0,j*} : g(y_{k_0,j} > 0)\}$. And note

$$\begin{aligned}
0 &< \sum_{j \in B_{k_0,j^*}} g^+(y_{k_0,j}) - \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}) + \sum j\delta_j \\
&< \sum_{j \in B_{k_0,j^*}} g^+(y_{k_0,j}) - \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}) + \frac{\varepsilon\delta}{10} \\
&< \sum_{j \in B_{k_0,j^*}} g^+(y_{k_0,j}) - \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}) + \frac{\delta}{5} \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}) \\
&< \sum_{j \in B_{k_0,j^*}} g^+(y_{k_0,j}) - (1 - \frac{\delta}{5}) \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}).
\end{aligned}$$

Thus $\sum_{j \in B_{k_0,j^*}} g^+(y_{k_0,j}) > (1 - \frac{\delta}{5}) \sum_{j \in B_{k_0,j^*}} f(y_{k_0,j}) \geq (1 - \frac{\delta}{5})(1 - \frac{\delta}{2}) \sum_{j \in B} f(y_{k_0,j}) > (1 - \delta) \sum_{j \in B} f(y_{k_0,j})$.

Let $N = \{(i,j) <_{r\ell} (\hat{i}, \hat{j}) : (i,j) \notin \{(k_0, j) : j \in B\}$ or $g(y_{i,j}) < 0\}$ and note

$$\sum_{(i,j) \in N} |g(y_{i,j})| \leq \sum_{j=1}^{\hat{j}} j\delta_j < \frac{\varepsilon\delta}{10} < \varepsilon$$

. □

Proof of Theorem 1.2. Assume $(x_{i,j})$ is monotone and basic (by renorming and passing to a subsequence). Apply Lemma 2.7 to $(x_{i,j})$ $\varepsilon = \delta/4$ and $\mathcal{F} = Ba(x^*)$ to yield the subarray $(x_{i,j})'$. Note

$$0 < \varepsilon < \frac{\delta}{2 + 2\delta}.$$

Let $(a_{i,j})_{(i,j) \in I} \in c_{00}(I)$ with $|a_{i,j}| \leq 1$ $k_0 \in \mathbb{N}$ and $F \subseteq \{j : |a_{k_0,j}| > \delta\}$. Let $f \in Ba(x^*)$ so that

$$\| \sum_{(k_0,j) \text{ where } j \in F} a_{i,j} x'_{i,j} \| = \sum_{(k_0,j) \text{ where } j \in F} a_{i,j} f(x'_{i,j}).$$

Define $F_+ = \{j \in F : a_{k_0,j} > 0$ and $f(x'_{i,j}) > 0\}$ and $F_- = \{j \in F : a_{k_0,j} < 0$ and $f(x'_{i,j}) < 0\}$. Thus

$$\sum_{(k_0,j) \text{ where } j \in F} a_{i,j} f(x'_{i,j}) \leq \sum_{(k_0,j) \text{ where } j \in F_+} a_{i,j} f(x'_{i,j}) + \sum_{(k_0,j) \text{ where } j \in F_-} a_{i,j} f(x'_{i,j}).$$

Assume without loss of generality that $\sum_{(k_0,j) \text{ where } j \in F} a_{i,j} f(x'_{i,j}) \leq 2 \sum_{(k_0,j) \text{ where } j \in F_+} a_{i,j} f(x'_{i,j})$. We may also assume $F \neq \emptyset$. Thus

$$\sum_{(k_0,j) \text{ where } j \in F_+} a_{i,j} f(x'_{i,j}) \geq \frac{1}{2} \| \sum_{(k_0,j) \text{ where } j \in F} a_{i,j} x'_{i,j} \| \geq \delta/2. \tag{1}$$

and so $\sum_{(k_0,j) \text{ where } j \in F_+} f(x'_{i,j}) \geq \delta/2$.

Thus by Lemma 2.7 there is a $g \in Ba(x^*)$ so that

$$\sum_{(k_0,j) \text{ where } j\in F_+} g(x'_{i,j}) \geq (1-\varepsilon) \sum_{(k_0,j) \text{ where } j\in F_+} f(x'_{i,j}),$$

and

$$\sum_{(i,j)\in J} |g(x'_{i,j})| \leq \varepsilon \sum_{(k_0,j) \text{ where } j\in F_+} f(x'_{i,j})$$

where $J = \{(i,j) \in I : (i,j) \leq_{r\ell} \max(\operatorname{supp}((a_{i,j}))$ and either $(i,j) \neq (k_0,j)$ for any $j \in F_+$ or $g(x'_{i,j}) < 0\}$. Thus

$$\begin{aligned}
\|\sum a_{i,j}x'_{i,j}\| &\geq g\left(\sum a_{i,j}x'_{i,j}\right) \\
&\geq \sum_{j\in F_+} a_{k_0,j}g(x'_{k_0,j}) - \sum_{(i,j)\in J} |a_{i,j}||g(x'_{i,j})| \\
&\geq \delta \sum_{j\in F_+} g(x'_{k_0,j}) - \sum_{(i,j)\in J} |g(x'_{i,j})| \\
&\geq \delta(1-\varepsilon) \sum_{j\in F_+} f(x'_{k_0,j}) - \varepsilon \sum_{j\in F_+} f(x'_{k_0,j}) \\
&= (\delta(1-\delta/4) - \delta/4) \sum_{j\in F_+} f(x'_{k_0,j}) \text{ since } \varepsilon = \delta/4 \\
&= (\delta/2 + \frac{\delta-\delta^2}{4}) \sum_{j\in F_+} f(x'_{k_0,j}) \geq \delta/2 \sum_{j\in F_+} f(x'_{k_0,j}) \text{ since } \varepsilon = \delta/4
\end{aligned}$$

Since $|a_{i,j}| \leq 1$ and by (1) we have

$$\sum_{j\in F_+} a_{k_0,j} f(x'_{i,j}) \geq \|\sum_{j\in F} a_{k_o,j}x'_{k_0,j}\| \geq \delta/2$$

So $\sum_{j\in F_+} f(x'_{i,j}) \geq \delta/2$. Therefore

$$\|\sum_{j\in F} a_{k_o,j}x'_{k_0,j}\| \geq 2\sum_{j\in F} f(x'_{k_0,j}) \geq 2(\frac{2}{\delta})\|\sum a_{i,j}x'_{i,j}\| = \frac{4}{\delta}\|\sum a_{i,j}x'_{i,j}\|.$$

$\square$

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