Dogs and Brachistochrones

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Contents

Abstract

The brachistochrone is a classic geometric shape that is the path of shortest time for a free rolling marble from one point to another. Here we examine a construction of that path using some Calculus techniques and Snell's law.

Keywords: Brachistochrone, Calculus of Variations, Snell's Law

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1 A Brachistochrone

What is the shortest distance between two points? Yes a straight line, but what is the shortest time between two points A and B where we build a path for a marble to roll from point A to point B starting with initial velocity of zero and only under the force of gravity.

So the marble will be moving very slowly early on in its trip. However, the marble will go faster and faster as rolls down the path. So which path seems like it might have the marble arrive first?

- path 1 - It does not seem like it is the fastest since the marble spends a lot of time going slow.

- path 2 - It could be the quickest path since it is the shortest distance.

- path 3 - It could be the quickest path. It is longer than path 2 but on this path the marble is going very fast right away and has a higher average speed than the marble on path 2.

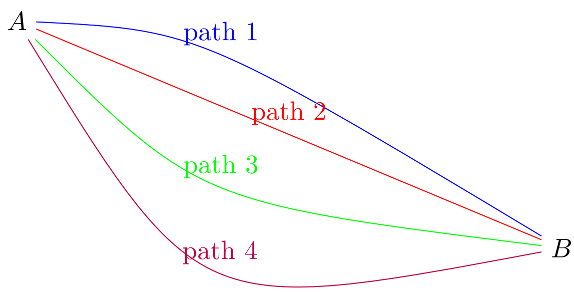

Fig. 1: I have drawn four paths from point A to point B. Notice path 2 is the straight line from point A to point B. Which path seems like it could be the quickest time?

- path 4 - This one could be the fastest too for a similar reasoning as for path 3.

Before we can answer this question we should know just how fast is the marble moving. Recall the acceleration due to gravity is $a(t) = -9.8\, m/s^2$, that the velocity is $v(t) = -9.8t\, m/s$ and that the height (position) of the marble is given by $y(t) = -4.9t^2\, m$. So solving the velocity and position equations for t yields

$$t = \frac{v}{-9.8} \text{ and } t = \sqrt{\frac{y}{-4.9}}.$$

So

$$\frac{v}{-9.8} = \sqrt{\frac{y}{-4.9}}.$$

Thus

$$v = -9.8\sqrt{\frac{s}{-4.9}} = \sqrt{19.6y} \approx 4.427\sqrt{s}. \quad (1)$$

Say $v = \sqrt{y}\, k$ where k is the constant 4.427 (notice I've already dropped the units). So when the ball is

- 1 meter below its initial position $v = \sqrt{1}k = 1k$
- 2 meter its initial position $v = \sqrt{2}k \approx= 1.4k$
- 3 meter its initial position $v = \sqrt{3} \approx= 1.7k$
- 4 meter its initial position $v = \sqrt{4} = 2k$
- 5 meter its initial position $v = \sqrt{5} \approx= 2.2k$

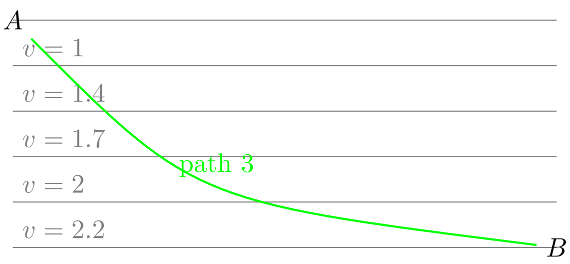

Fig. 2: As the marble rolls down the path we can see it is rolling at 2.2 k which is much faster than near the top.

2 Euler-Lagrange Differential Equation

Our goal is to find the path that minimizes time. Sounds like a Calculus of variations problem to me! To maximize the functional

$$I(y) = \int_a^b f(x, y, y')dx$$

we recall the Euler-Lagrange formula

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

And its simplification, Beltrami's formula

$$y'\frac{\partial f}{\partial y'} - f = c$$

where c is a constant.

Don't you remember this formula! Don't remember your calculus of variations either! [1] Well we need to do something else then.

3 Dogs

I have a dog named Bella and she enjoys fetching the stick. So it is another path problem. When I throw the stick to her she doesn't just run a straight path to the stick, she minimizes the the time of her path. Sounds close enough to what we were doing. So let's talk about Bella fetching the stick.

When I throw the stick to her on flat land she runs straight at the stick. But when I throw the stick to her at the edge of a lake she makes a turn. She travels a straight line on the land where she runs fast and then makes turn as she hits the water (she swims much slower than she runs).

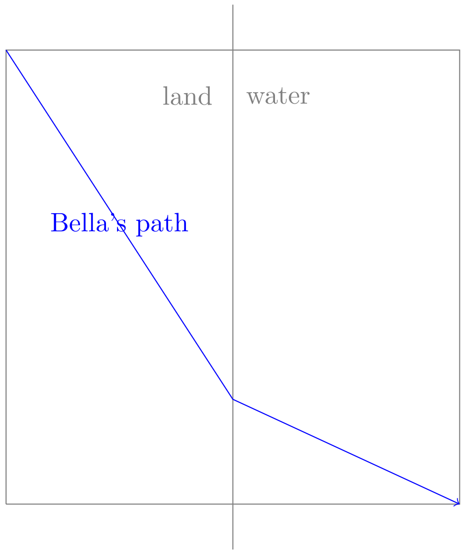

Fig. 3: She runs about 6 meters per second and swims about 3 meters per second. So she angles a bit to remain at her fast speed on land before swimming after the stick.

Bella knows calculus [Pennings(2003)]. She can minimize the time. How does she do it? Let's calculate it the way Bella does. We know $D = RT$. That is distance equals rate times time.

So we have (see Figure 3)

$$\begin{aligned} D_1 &= R_1T_1 & D_2 &= R_2T_2 \\ D_1 &= 6T_1 & D_2 &= 3T_2 \\ \sqrt{x^2 + 10^2} &= 6T_1 & \sqrt{(20-x)^2 + 10^2} &= 3T_2 \end{aligned}$$

Thus

$$T_1 = \frac{\sqrt{x^2 + 10^2}}{6}$$

[1] If you do know something about PDE's and calculus of variations we will solve in the Section 6.1.

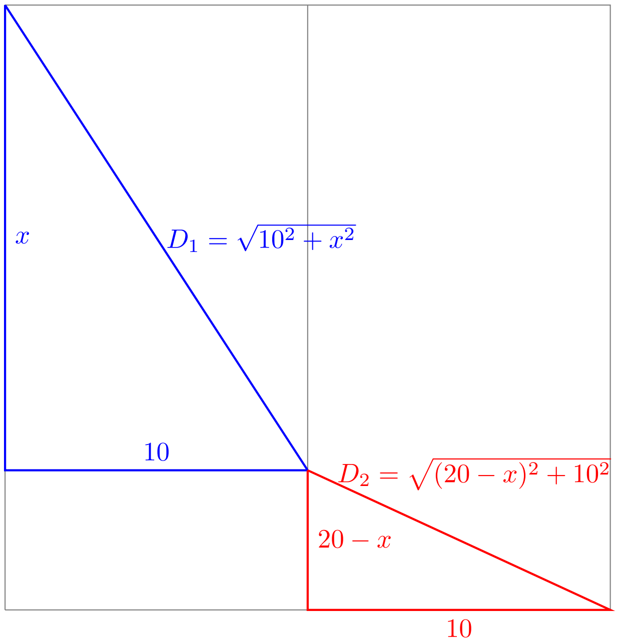

Fig. 4: We use the Pythagorean Theorem to obtain the equations for D_1 and D_2.

and

$$T_2 = \frac{\sqrt{(20-x)^2 + 10^2}}{3}.$$

When Bella tries to minimize the time traveled she minimizes the equation

$$T = T_1 + T_2 = \frac{\sqrt{x^2 + 10^2}}{6} + \frac{\sqrt{(20-x)^2 + 10^2}}{3}$$

We have two techniques to minimize functions, we could graph and estimate the minimum point from the graph or we could use Calculus. Let's go through both below.

3.1 Graph and Estimate Minimum

We graphed the function $T(x)$ see Figure 5 and the minimum appears to be at about $x = 15.383$ meters.

From the figure we see that Bella should aim at a point at about where $x \approx 15.383$ meters.

3.2 Use Calculus to compute the Minimum

To use Calculus we want to minimize the function

$$T(x) = \frac{\sqrt{x^2 + 10^2}}{6} + \frac{\sqrt{(20-x)^2 + 10^2}}{3}.$$

First we take the derivative and set it equal to zero to get

$$T'(x) = \frac{x}{6\sqrt{x^2 + 10^2}} - \frac{20-x}{3\sqrt{(20-x)^2 + 10^2}} = 0.$$

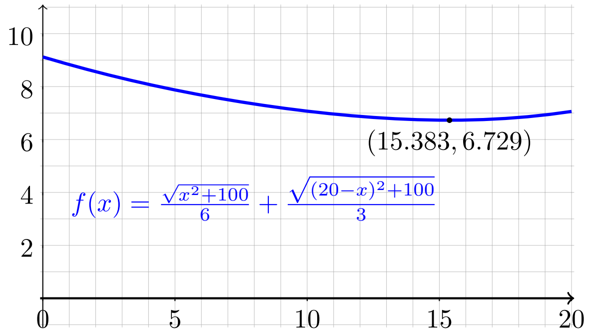

Fig. 5: Here we graph the function $T(x)$. We see that the minimum T occurs when $x \approx 15.383$ and the minimum time is about 6.729 seconds.

Then we get

$$\frac{x}{6\sqrt{x^2 + 10^2}} = \frac{20-x}{3\sqrt{(20-x)^2 + 10^2}}.$$

By cross multiplying and factoring (which includes solving a quartic equation) we get $x \approx 15.383$ again.

Another way to solve this could be to use Snell's Law. But my guess is that Bella uses the Calculus method.

4 Snell's Law

Snell's Law (named after Willebrord Snellius) is the optics law that we use to determine how much light bends when traveling through different media. for example light bends when going from air to water or from air to glass.

Snell's Law

Let θ_1, θ_2 be the angles of incidence as in Figure 6 and let v_1, v_2 be the velocities of light through the two media. Then

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}.$$

Snell's Law tells us how light bends, however, Snell's law also can be interpreted a different way. The light bends to minimize the time from it's starting point to its finishing point (see Section 6.2).

We could use Snell's law to calculate the minimum time to go from point A to point B. We would know that the

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{6}{3} = 2.$$

And with a little bit of algebra and trigonometry we should arrive at the same solution as before.

4.1 Bella on a different Path

What if Bella travels through several terrains to fetch the stick. Let's say she travels through open

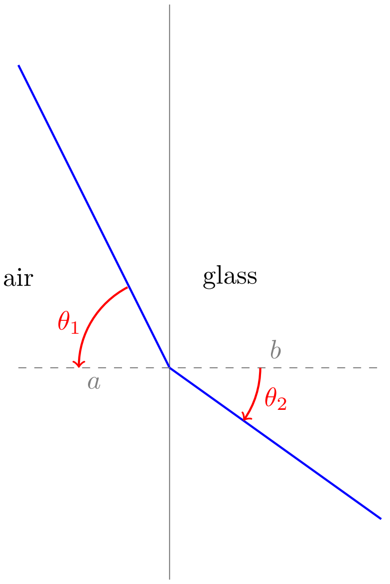

Fig. 6: The blue line is the path of light as travels from air into glass. The bending of the light (refraction) is something you witness in your everyday life.

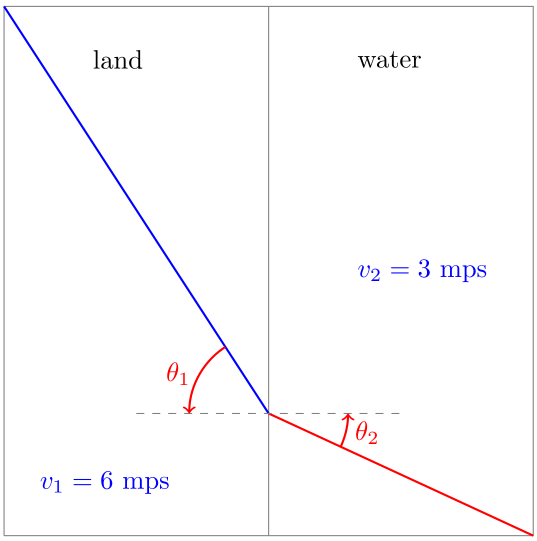

Fig. 7: We use Snell's Law.

field, thickets, hills, pond, swamp with speeds given below in meters per second.

open field	thickets	hills	pond	swamp
2.2	2.0	1.7	1.4	1.0

Wait a minute something looks familiar! Compare Figure 2 to Figure 8. We have just computed an approximation of the Brachistochrone.

5 Questions

Some discussion questions.

1. Recall Snell's Law $\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}$. Let's apply it for two mediums with $v_1 = c$ (air) and

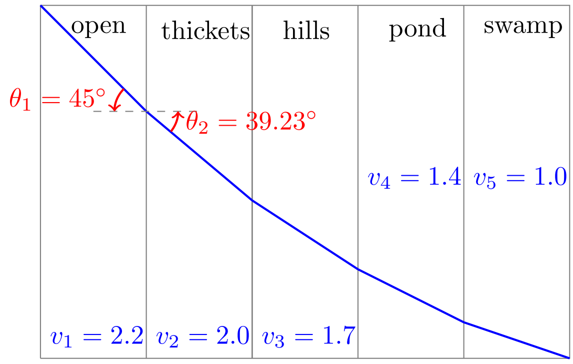

Fig. 8: We use Snell's Law.

$v_2 = 0.75c$ (water) and the angle $\theta_2 = 70°$. So Snell's law yields

$$\frac{\sin(\theta_1)}{\sin(70°)} = \frac{c}{.75c}$$

$$\sin(\theta_1) = \frac{4}{3}\sin(70°) \approx 1.25$$

But we know $-1 \leq \sin(\theta_1) \leq 1$. What is happening?

2. What is the shape of the Brachistochrone? The result from Section 6.1 gives us a formula for a cycloid. What is a cycloid?

3. What is a tautochrone? And what does it have to do with the Brachistohrone?

4. Who is Johann Bernoulli?

6 Appendices

6.1 Eine Kleine Calculus of Variations

The main problem under consideration in Calculus of Variations is finding extrema for real valued functions of functions.

Let $C^2(\mathbb{R})$ be the set of twice differentiable functions on $\mathbb{R}$. Let $I : C^2(\mathbb{R}) \to \mathbb{R}$. We are looking for the extrema of I.

The basic example is

$$I(y) = \int_A^B f(x, y, y')dx \tag{2}$$

where y is a twice differentiable function of x. When $f(x, y, y')$ is independent of x then our extrema satisfies the Beltrami equation

$$y'\frac{\partial f}{\partial y'} - f = c \text{ where c is a constant.} \tag{3}$$

So given Equation 2 our extrema (minimums in our problem) are functions that satisfy the Beltrami equation. So we will first find the integral equation and the use the Beltrami equation to the the function which is minimal.

So our brachistochrone problem has $I(y)$ being the time for a marble to travel the path defined by the curve y. Recall we had two points, say $A = (x_0, y_0)$ and $B = (x_1, y_1)$. We have already seen that $v = \sqrt{2g(y_0 - y)}$ see Equation 1 and where $g = -9.8$ is the force of gravity. Since $v = \frac{ds}{dt}$ where s is the arc length traveled by the marble we have $dt = \frac{1}{v}ds$. Now we can integrate from points A to B to get

$$
\begin{aligned}
t &= \int_A^B dt = \int_A^B \frac{1}{v} ds \\
&= \int_A^B \frac{1}{v}\sqrt{1+(y')^2}dx \\
&\text{where } ds = \sqrt{1+(y')^2}dx \\
&\text{is from a standard calculus textbook} \\
&= \int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{2g(y_0-y)}}dx \\
&\text{where } v = \sqrt{2g(y_0-y)} \text{ from above} \\
&= \frac{1}{\sqrt{2g}}\int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{y_0-y}}dx \\
&\text{where } v = \sqrt{2g(y_0-y)} \text{ from above.}
\end{aligned}
$$

So for our problem we have

$$
I(y) = \frac{1}{\sqrt{2g}}\int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{y_0-y}}dx
$$

where the function y that has minimum value satisfies the Beltrami equation with $f(x, y, y') = \frac{\sqrt{1+(y')^2}}{\sqrt{y_0-y}}$. Now we substitute f into the the Beltrami equation and solve.

$$
y'\left(\frac{y'}{\sqrt{y_0-y}\sqrt{1+(y')^2}}\right) - \frac{\sqrt{1+(y')^2}}{\sqrt{y_0-y}} = c
$$

now we solve for y' then we will integrate to get our function y.

$$
\begin{aligned}
\frac{(y')^2 - (1+(y')^2)}{\sqrt{y_0-y}\sqrt{1+(y')^2}} &= c \\
\frac{-1}{\sqrt{y_0-y}\sqrt{1+(y')^2}} &= c \\
(y_0-y)(1+(y')^2) &= \frac{1}{c^2} \\
1+(y')^2 &= \frac{1}{c^2(y_0-y)} \\
(y')^2 &= \frac{1}{c^2(y_0-y)} - 1
\end{aligned}
$$

Thus

$$
\begin{aligned}
(y')^2 &= \frac{1}{c^2(y_0-y)} - 1 = \frac{1-c^2(y_0-y)}{c^2(y_0-y)} \\
y' &= \sqrt{\frac{1-c^2(y_0-y)}{c^2(y_0-y)}} \\
\frac{dy}{dx} &= \sqrt{\frac{1-c^2(y_0-y)}{c^2(y_0-y)}}.
\end{aligned}
$$

So our solution satisfies

$$
x = \int dx = \int \sqrt{\frac{c^2(y_0-y)}{1-c^2(y_0-y)}}\,dy.
$$

Integrating this is quite straightforward using the substitution $y - y_0 = \frac{1}{c^2}\sin^2(t/2)$ and the trigonometric identity $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$. After the substitution we get

$$
\begin{aligned}
x &= \frac{1}{2c^2}[t - \sin(t)] + b \text{ where b is a constant} \\
y &= y_0 + \frac{1}{c^2}\sin^2(t/2) \\
&= y_0 + \frac{1}{c^2}\left(\frac{1}{2}(1-\cos(t))\right) \\
&= y_0 + \frac{1}{2c^2}(1-\cos(t)).
\end{aligned}
$$

Thus in parametric form or equation is

$$
\begin{aligned}
x &= x_0 + \frac{1}{2c^2}[t - \sin(t)] \\
y &= y_0 + \frac{1}{2c^2}(1-\cos(t)).
\end{aligned}
$$

Two observations

1. This shape is a well known shape called a cycloid. It maybe worth it to you to look it up.
2. Notice gravity is absent from our formula. How would the equation change if we ask for the brachistocrone on the moon?

6.2 Snell's Law is minimizing time

$$
\begin{aligned}
T &= T_1 + T_2 \\
T(x) &= \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{(x-c)^2+b^2}}{v_2}
\end{aligned}
$$

Next we compute the derivative and set it to zero

$$
T(x) = \frac{x}{v_1\sqrt{x^2+a^2}} - \frac{x-c}{v_2\sqrt{(x-c)^2+b^2}} = 0.
$$

So $\frac{x}{v_1\sqrt{x^2+a^2}} = \frac{x-c}{v_2\sqrt{(x-c)^2+b^2}}$. Thus

$$
\frac{\frac{x}{\sqrt{x^2+a^2}}}{v_1} = \frac{\frac{x-c}{\sqrt{(x-c)^2+b^2}}}{v_2}.
$$

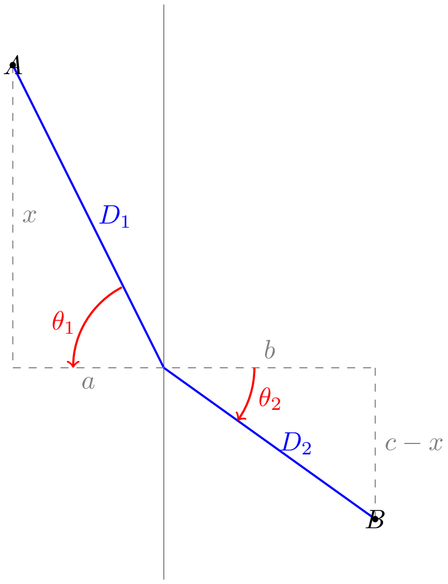

Fig. 9: The blue line is the path of light as travels from one medium, say air, into another medium, say water. You can see the refraction (ie bending) of the light as it passes from one medium to the next. We use the Pythagorean Theorem to obtain $D_1 = \sqrt{a^2 + x^2}$ and $D_2 = \sqrt{(c-x)^2 + b^2}$. Also note $\sin(\theta_1) = \frac{a}{\sqrt{a^2+x^2}}$ and $\sin(\theta_1) = \frac{b}{\sqrt{(c-x)^2+b^2}}$.

Notice from the Figure 9 that

$$\sin(\theta_1) = \frac{x}{\sqrt{x^2 + a^2}}$$

and

$$\sin(\theta_2) = \frac{x - c}{\sqrt{(x-c)^2 + b^2}}.$$

Therefore the x value that minimizes the time spent traveling by the light is $\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}$.

Fig. 10: Here Bella is hanging out with the chickens and one duck.

References

[Komzsik(2014)] Louis Komzsik. *Applied calculus of variations for engineers.* CRC Press, 2014.

[Pennings(2003)] Timothy J Pennings. Do dogs know calculus? *College mathematics journal*, 34(3):178–182, 2003.