# DISTANCES

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Abstract. We have many ways to determine distance. One is to use an old formula $D = \sqrt{x^2 + y^2}$ with roots that include the Pythagorean theorem. We also can find our way through a gridded city with the taxicab metric and used simple ear tests to determine if a song (that has been compressed) sounds right. That is, we are estimating the distance from a song streamed to the original song.

What is a "distance?" We will examine the abstract idea of a distance, called a norm, and explore a odd norms few resulting from our "abstract idea." Does it still make sense as a distance?

## 1. Some Algebra Warm Up

First we should start with a few sketches to test our algebra skills. Can anybody sketch the following $x^2 + y^2 = 1$? What is this shape and tell me what you remember about the formula. This graph is called the unit circle. That is a circle with radius 1 (unit in unit circle tells us the radius is one).

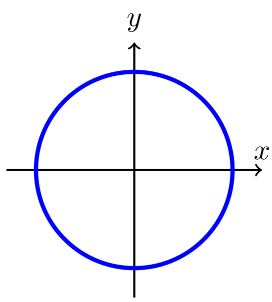


Figure 1. The graph of $x^2 + y^2 = 1$, the unit circle.

I would to start today with is an exploration of slight alterations to the unit circle. Let's draw $|x|^1 + |y|^1 = 1$, $|x|^3 + |y|^3 = 1$, $x^4 + y^4 = 1$, $x^1.5 + y^1.5 = 1$, and $x^0.5 + y^0.5 = 1$. Maybe we can even try to see what happens if the exponent gets really large like $|x|^5 + |y|^5 = 1$, $|x|^{10} + |y|^{10} = 1$, and $|x|^\infty + |y|^\infty = 1$.

Graph the following.

(1) $|x|^1 + |y|^1 = 1$
(2) $|x|^3 + |y|^3 = 1$
(3) $x^1.5 + y^1.5 = 1$
(4) $|x|^0.5 + |y|^0.5 = 1$ [1]

Now graph the following as the exponent gets large and try to guess the last one.

[1]Note $|x|^0 + |y|^0 = 1$ doesn't quite make sense. Why?

(1) $|x|^5 + |y|^5 = 1$
(2) $|x|^{10} + |y|^{10} = 1$
(3) $|x|^\infty + |y|^\infty = 1$

One last problem to graph for fun at home.

(1) $|x|^{2.5} + |y|^{2.5} = 1$
(2) $|\frac{x}{6}|^{2.5} + |\frac{y}{5}|^{2.5} = 1$ [2]

Now that we are done with our warm up, on to the lesson on distances.

## 2. Distance from point to point

We have seen how to compute the distance between two points before. In mathematics we have many notions of distance and the one we are all most familiar with is called te Euclidean distance[3]. Let's remind ourselves how to compute a distance.

Define the points as follows $A(1, -2)$, $B(4, 3)$, and $O(0, 0)$. The distance between the points A and B we write and compute as follows

$$D(A, B) = D((1, -2), (4, 3)) = \sqrt{(1-4)^2 + (-2-3)^2} = \sqrt{(-3)^2 + 5^2} = \sqrt{34}.$$

$$D(O, B) = D((0, 0), (4, 3)) = \sqrt{(4)^2 + (3)^2} = \sqrt{25} = 5.$$

We sometimes write a distance from an origin to a point as the "double bar" norm $D(O, B) = \|B\|$ where we think of $B$ as a vector.

If you have not seen a vector just think of it as a point with a few extra properties. Two of those properties are multiplication and addition. So we can compute the following

- multiplication of a vector by a number
$$3A = 3(1, -2) = (3, -6)$$
- addition of two vectors
$$A + B = (1, -2) + (4, 3) = (5, 1)$$

So to compute the following $\|3A\|$, and $\|A + B\|$ we would first compute the vector then use the norm (distance from the origin) measure.

- multiplication of a vector by a number $\|3A\| = \|3(1, -2)\| = \|(3, -6)\| = \sqrt{3^2 + (-6)^2} = \sqrt{9 + 36} = \sqrt{45}$
- addition of two vectors $\|A + B\| = \|(1, -2) + (4, 3)\| = \|(5, 1)\| = \sqrt{5^2 + 1^2} = \sqrt{25 + 1} = \sqrt{26}$

To visualize these operations we view multiplication as scaling. So the vector $A$ is an arrow from $O$ to $A$ and $3A$ is an arrow from $O$ to $3A$.

We visualize the addition of vectors as travelling the first arrow and then travelling the second arrow to result in a new arrow.

Some Properties of distance. I mentioned that I would expect the norm of $2B$ to be twice that of $B$. There is some ogic to that. This poperty is called **scalability** and we write as

$$\|kB\| = |k|\|B\|.$$

[2]See Piet Hein and Sergels torg.

[3]It is called the Euclidean distance, clearly, after Euclid.

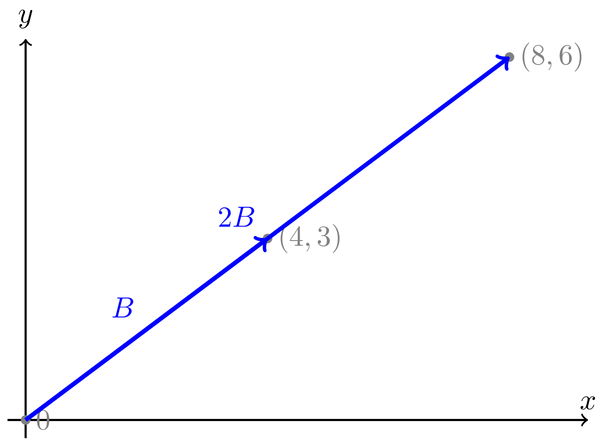


FIGURE 2. Here we draw $B = (4, 3)$ and $2B = (8, 6)$. And as the picture seems to display the length of $2B$ is twice that of $B$. Note $\|B\| = \|(4, 3)\| = 25$. What is the norm (length) of $2B$? Is it 50?

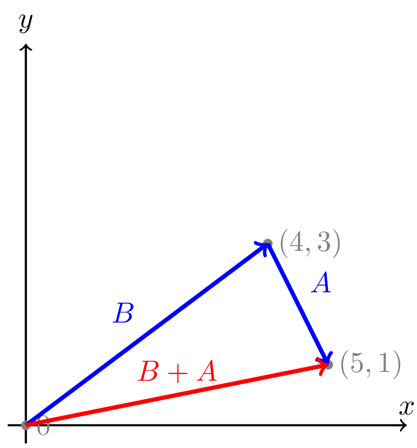


FIGURE 3. To demonstrate $B + A$ we travel the length of $B$ first then starting from the end of $B$ we travel the length of $A$. Note $B + A = (5, 1)$. So the resulting vector of $B + A$ is drawn in red. Would it matter if we travelled the path of A first and then B? Can you draw that?

where $k$ is any number and $B$ is any vector. Note we need the absolute value around $k$ in case $k$ is negative.

We also have a property called the **triangle inequality**. Looking at the below what inequality symbol goes in place of the question mark?

$$\|B\| + \|A\| ?? \|B + A\|.$$

There are many properties we can have for a distance. These are two. Some mathematical distances don't require the linear scalability. Norms do.

**Definition.** Let $V$ be a vector space over the real numbers. Let $\| \cdot \|$ be a function from $V$ to the non-negative real numbers. For all vectors $A$ and $B$ in $V$ and all real numbers $k$ if

(1) $\|A\| = 0$ if and only if $A = 0$,
(2) $\|kA\| = |k|\|A\|$, and
(3) $\|A + B\| \leq \|A + B\| + \|A + B\|$

Then we say that $\|\cdot\|$ is a bf norm.[4]

We have been working in 2-dimensions, $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$. But we can work in 3-dimensions ($\mathbb{R}^3$), 4-dimensions ($\mathbb{R}^4$) or even infinitely many dimensions ($\mathbb{R}^\infty$). The calculations are quite similar.

$$\|(1, 2, 3)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$
$$\|(1, 2, 3, 4)\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}$$
$$\|(1, \frac{1}{2}, \frac{1}{3}, \cdots)\| = \sqrt{1^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + \cdots} = \frac{\sqrt{6}}{\pi}$$

Well the first two seem quite straight forward. I think the third one will take a little calculus to see.

## 3. Make up your own norms

This is the Institute for **Creative** Problem solving. So let's get creative. We have defined one norm so far (keep in mind it is an idea of distance).

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

We call this the Euclidean norm or the $\ell_2$ norm and it is sometimes written as $\|(x, y)\|_2$. Can you define a new norm (let's stick with $\mathbb{R}$ for simplicity)? Try and create your own and we will test whether or not your creation is a norm.

Here are a few I came up with.

(1) $\|(x, y)\| = |x| + |y|$
(2) $\|(x, y)\| = x + y$
(3) $\|(x, y)\| = \sqrt{|x| + |y|}$
(4) $\|(x, y)\| = \sqrt[4]{x^4 + y^4}$
(5) $\|(x, y)\| = \sqrt[10]{x^{10} + y^{10}}$
(6) $\|(x, y)\| = (x^{1.5} + y^{1.5})^{1/1.5}$
(7) $\|(x, y)\| = (x^{0.5} + y^{0.5})^2$
(8) $\|(x, y)\| = (x^0 + y^0)^\infty$
(9) $\|(x, y)\| = (x^\infty + y^\infty)^0$

Let's test if any of the above are norms. We will need to test the three rules

Some notes from 3a here

## 4. Other Unit Circles

## 5. Other Norms

Here are a few other norms of interest.

$\ell_1$ **norm.** $\|(x, y)\|_1 = |x| + |y|$.

[4] What is a vector? What is a real number?

**$\ell_\infty$ norm.** $\|(x,y)\|_\infty = \max\{|x|,|y|\}$.

**Summing Norm.** $\|(x,y)\|_{SUM} = \max\{|x|,|x+y|\}$.

**Taxicab Norm.** Imagine a grid and driving a taxi. You can't drive the taxi diagonally. You can only drive east and west or north and south. The taxicab norm is the shortest distance between two points driving in this manner. Look at the Figure 4. So the $\|(6,-3)\| = 9$. This norm should look familiar. What is it?

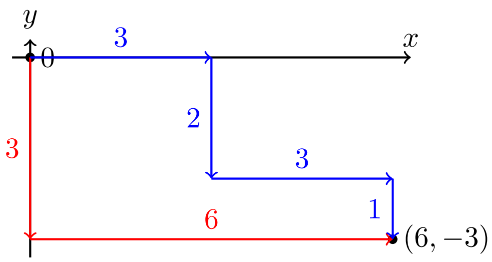


FIGURE 4. To drive from the origin to the point $(6,-3)$ there are several routes. In fact several shortest routes. I get 9 as the shortest. The blue path is $3+2+3+1=9$, and the red path is $3+6=9$.

All of these are norms and a few are repeats.
Can you

(1) Draw the unit circle for the summing norm.
(2) Prove the summing norm is a norm. That is show it satisfies the three rules, zero is zero, scalability and the triangle inequality.

## 6. Volumes of various $\ell_p$-norm Balls

From [1].

$$V = 2^n \frac{\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}$$

We have primarily been working in the 2-dimensional cas. So $n=2$. Thus the area of the $\ell_p$-norm circles is

$$A = 2^2 \frac{\Gamma(1+\frac{1}{p})^2}{\Gamma(1+\frac{2}{p})} = 4\frac{\Gamma(1+\frac{1}{p})^2}{\Gamma(1+\frac{2}{p})}. \tag{1}$$

The gamma function, $\Gamma(x)$ may look unfamiliar, but it is not to odd if we just consider positive whole numbers as inputs. Fo $x \in \mathbb{N}$ we have

$$\Gamma(x) = (x-1)!$$

And the gamma function "fills in" the values between the whole numbers for a factorial function. For now let's just use a calculator.

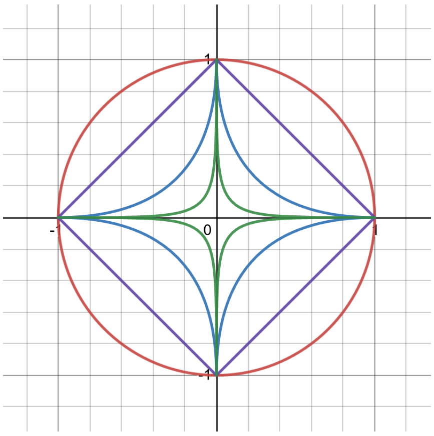

FIGURE 5. The graph of $x^2+y^2=1$, $|x|^1+|y|^1=1$, $|x|^{0.5}+|y|^{0.5}=1$, $|x|^{0.3}+|y|^{0.3}=1$ ranging in colors from red, blue, green and then purple. Guess what $|x|^0+|y|^0=1$ could look like at a limit.

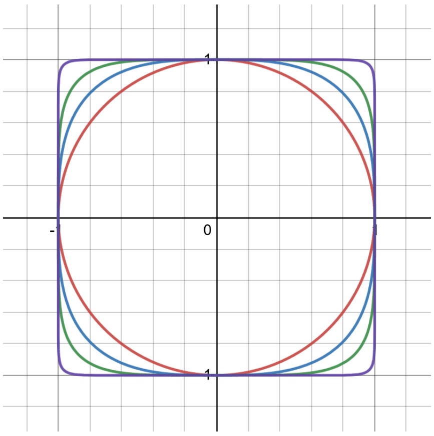

FIGURE 6. The graph of $x^2+y^2=1$, $x^3+y^3=1$, $x^5+y^5=1$, and$x^{20}+y^{20}=1$, ranging in colors red, blue, green and then purple. Guess what $|x|^\infty+|y|^\infty=1$ could look like at a limit.

## 7. Some answers

## References

[1] Wang, Xianfu. "Volumes of generalized unit balls." *Mathematics Magazine* 78.5 (2005): 390-395.

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