Math 3160 - Test 2 Review

1 Vectors

1. Find a unit vector that when positioned at the origin forms a 30° angle with the x-axis.

2. Find a vector of length three units that when positioned at the origin forms a 240° angle with the x-axis.

3. Let $\mathbf{v} = (1, 3, 4)$ and $\mathbf{w} = (1, -1, 0)$ be vectors in $\mathbb{R}^3$. and let $P(1, 1, 1)$ and $Q(0, -4, 0)$ be two points in $\mathbb{R}^3$.

   (a) Find a vector that is parallel to $\mathbf{v}$ and unit.

   (b) Compute $\|2\mathbf{v} - \mathbf{w}\|$.

   (c) Compute the angle between $\mathbf{v}$ and $\mathbf{w}$.

   (d) Find the equation of a plane containing P and with normal vector $\mathbf{v}$.

4. Find the parametric equations for the line (in $\mathbb{R}^3$) so that

   (a) the line contains the point $P(0, 1, 2)$ and $Q(1, 2, 3)$.

   (b) the line contains the point $P(-5, 1, 3)$ and is parallel to the vector $(1, 2, 3)$.

5. Find equation (both the normal equation and the parametric equation) for the plane (in $\mathbb{R}^3$) so that

   (a) the plane contains the points $P(2, 2, 2)$, $Q(1, 2, 3)$ and $R(1, -1, 0)$.

   (b) the plane contains the origin and is perpendicular to the vector $(1, 2, 3)$.

   (c) the plane contains the point $(1, 2, 3)$ and contains the vectors $\mathbf{v} = (1, 0, -3)$ and $\mathbf{w} = (2, 1, 3)$.

6. Define the planes P_1 and P_2 as follows:

$$P_1 : x - 2y + z = 12$$

$$P_2 : 3x - 3y + z = 4$$

   (a) What are the two normal vectors for the above planes.

(b) Find the angle between the two above planes.

(c) Find two points one on each of the above planes.

(d) Find set of all points that lay in both planes.

(e) The two planes intersect in a line, find the parametric equation of that line.

7. Define the planes P_1 and P_2 as follows:

$$P_1 : x - 2y + z = 12$$

$$P_2 : 3x - 6y + 3z = 4$$

Show the planes are parallel. Do they intersect? Find the solution set to their intersection.

8. Let $P(1, 0, 2, 0)$, $Q(0, 0, 2, -1)$, $R(0, 1, 1, 0)$ and $S(1, 1, 1, 0)$ be points in $\mathbb{R}^4$. Find the equation of the hyperplane (both the normal equation and the) that contains these four points.

9. For the hyperplane (in $\mathbb{R}^5$) given below: find its normal vector and find its parametric equation.

$$x_1 + -4x_3 + 2x_4 + x_5 = 3$$

2 Vector Spaces and Subspaces

10. Let $V = \mathbb{R}^3$ equipped with usual vector addition and scalar multiplication. Prove V is a vector space. That is, prove all 10 Axioms.

11. Let $V = \mathbb{R}^2$. And define the two operations

$\oplus$: $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$

$\odot$: $k \odot (x_1, y_1) = (kx_1, ky_1)$

(a) Compute $(0, 4) \oplus (-2, 3)$ and compute $2 \odot (1, 1)$.

(b) Show $\mathbf{0} \neq (0, 0)$.

(c) Show $\mathbf{0} = (-1, -1)$.

(d) Prove Axiom 5. That is, for each $\mathbf{v}$ find $-\mathbf{v}$ so that

$$\mathbf{v} \oplus -\mathbf{v} = \mathbf{0}.$$

(e) V does satisfy some of the vector space axioms, but not all of the axioms. Find two axioms that fail.

12. State the two step subspace test.

13. Let $W = \{(a,b,c) \in \mathbb{R}^3 : \text{ where } a+b+c=1\}$.

    (a) Use the two step subspace test to show $(W,+,\cdot)$ is a subspace.

    (b) What geometric shape is W? Hint I gave it in standard form.

    (c) Give me the parametric equation for the geometric object defined in the set W.

14. Let $W = \{(x,y,z) \in \mathbb{R}^3 : \text{ where } x-3y-z=0\}$.

    (a) Use the two step subspace test to show $(W,+,\cdot)$ is a subspace.

    (b) What geometric shape is W? Hint I gave it in standard form.

    (c) Give me the parametric equation for the geometric object defined in the set W.

3 Linear Independence

15. Let $S = \{(1,2,1),(0,1,2),(0,-1,0)\}$.

    (a) Is S linearly independent? (There is an easy test for this problem).

    (b) Is $(2,2,2) \in \text{Span}(S)$? If yes what is a linear combination of the vectors in S that equals $(2,2,2)$?

    (c) Does S span $\mathbb{R}^3$?

16. Let $S = \{x, x+2, x^3-x-1, x^3\}$ be a set in P_3. Use $v_1 = (0,1,0,0)$, $v_2 = (2,1,0,0)$, $v_3 = (-1,-1,0,1)$, and $v_4 = (0,0,0,1)$.

    (a) Is S linearly independent?

    (b) Is $x^3+x^2+x+1 \in \text{Span}(S)$? If yes what is a linear combination of the polynomials in S that equals x^3+x^2+x+1?

    (c) Is $4x^3-2x \in \text{Span}(S)$? If yes what is a linear combination of the polynomials in S that equals $4x^3-2x$?

    (d) Does S span P_3?

4 Span, Basis

17. Let $B = \{(1,2,1),(0,1,2),(0,-1,0)\}$.
    (a) Is B a basis for $\mathbb{R}^3$
    (b) Write the vector $(1,0,-1)$ relative to the basis B.
    (c) Write the vector (a,b,c) relative to the basis B.
    (d) Find the change of basis matrix from the standard basis to the basis B. (we called it $P_{\text{STANDARD}\to B}$ in class).

18. For the following system of linear equations.

$$\begin{array}{ccccccc} 2x_1 & -2x_2 & +4x_3 & & -6x_5 & = 2 \\ & & x_3 & +6x_4 & & = 0 \end{array}$$

    (a) Find the solution set.
    (b) Find a basis for the solution set.
    (c) What is the dimension of that solution set?

19. For the following subspace of P_3

$$W = \{a + bx + cx^2 + dx^3 : a = -c \text{ and } b = c + d\}$$

    (a) Find a basis for W.
    (b) What is the dimension of that solution set?

5 Change of Basis Matrix

20. ~~Let $B = \{(1,0),(0,1)\}$, $B_1 = \{(-1,1),(2,3)\}$, and $B_2 = \{(1,-1),(1,1)\}$.~~
    (a) ~~Find the change of basis matrices for $P_{B_1\to B_2}$ and $P_{B_1\to B_2}$.~~
    (b) ~~Find the coordinates of the point $(4,6)$ (given in the standard basis) relative to the bases B_1 and B_2.~~
    (c) ~~Find the change of basis matrices for $P_{B\to B_2}$ and $P_{B_2\to B}$.~~
    (d) ~~Find the coordinates of the point $(2,-4)$ (given in the standard basis) relative to the bases B and B_2. Graph this point the two separate coordinate axes B and B_2.~~

6 Row Space, Column Space & Null space

21. Let W be the plane $x - 2y + z = 0$ in $\mathbb{R}^3$.

    (a) Find the parametric equation for the plane.

    (b) Find a basis for W.

    (c) Compute the solution set to the linear system $x - 2y + z = 0$ in $\mathbb{R}^3$.

22. Let W be the hyperplane $x_1 - 2x_2 + x_3 + 6x_4 = 0$ in $\mathbb{R}^4$.

    (a) Find the parametric equation for the hyperplane.

    (b) Find a basis for W.

    (c) Compute the solution set to the linear system $x_1 - 2x_2 + x_3 + 6x_4 = 0$ in $\mathbb{R}^4$.

23. Let $A = \begin{bmatrix} -1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & -1 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{bmatrix}$.

    (a) Find a basis for the Column Space of A, COL(A), and the row space of A, ROW(A).

    (b) Compute the dimension of COL(A) and ROW(A).

    (c) Find a basis for the null space of A, NULL(A).

    (d) Compute the dimension of NULL(A).

24. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the formula $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ y+z \\ x-z \end{bmatrix}$.

    (a) Find the matrix, A, to represent the linear transformation T.

    (b) Compute the basis for the Range of T, which is the Column Space of A.

    (c) Find a basis for the null space of A, NULL(A).

    (d) Compute the dimension of COL(A) and NULL(A). The dimension of the range of T is called the rank of T and the dimension of the null space is called the nullity.

    (e) What is the dimension of the domain of T and the codomain of T? Again, compare Rank, Nullity and the dimension of the Domain. Do you see a relation?

7 Basic Transformations

25. Write the matrix for the following transformations described below.

    (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is rotated by $45°$ counter-clockwise.

    (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is reflected about the x-axis.

    (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the x-axis is contracted by half and the y-axis is dilated by 2.

    (d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is rotated by $30°$ counter-clockwise and then reflected about the x-axis.

    (e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is reflected about the x-axis and then rotated by $30°$ counter-clockwise.

    (f) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where the x-axis is contracted by half and the z-axis is dilated by 2.

8 Eigenvalues, Eigenvectors and Diagonalization

26. ~~For the following matrices find the characteristic equation, the eigenvalues and their corresponding eigen vectors.~~

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

and

$$E = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix},$$

27. ~~for the above matrices, determine if they are diagonalizable. State why or why not. And if it is diagonalizable, diagonalize it. That is, find P and D.~~

28. ~~Diagonalize the matrix below.~~

$$\begin{bmatrix} 4 & 0 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$