Math 6250: Test 1 Review

1 Foundations

1. State the Peano Axioms.
2. Use induction to prove something simple.
3. Define equivalence relation.
4. Be able to show a given relation is an equivalence relation.
5. Be able to show a given relation with an operation is well defined. Know what well defined means.
6. Define $(\mathbb{Z}, +, \cdot)$ using equivalence relation and show the operations are well defined.
7. Define $(\mathbb{Q}, +, \cdot)$ using equivalence relation and show the operations are well defined.

2 Functions and Cardinality

8. Know the definition of injective and surjective.
9. Prove a statement like: If f is injective and g is injective then $f \circ g$ is injective.
10. Show a given function is injective or surjective. For the folllowing determine if the function is injective or is not injective and prove. Also determine if the function is surjective or not and prove.
    (a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x - 3$
    (b) $f : (1, 2) \to (-2, 4)$ where $f(x) = 2x - 3$
    (c) $f : (1, 2) \to (-1, 3)$ where $f(x) = 2x - 3$
    (d) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$
    (e) $f : (-\infty, 0) \to \mathbb{R}$ where $f(x) = x^2$
11. Prove $\mathbb{Q} \sim \mathbb{N}$, $\mathbb{Z} \sim \mathbb{N}$, $\mathbb{R} \not\sim \mathbb{N}$,

3 The Real and the Complex Numbers

12. State the definition of the Reals.

13. State the definition of a Field.

14. Compute a sup or inf.

15. Prove: Let $\alpha = \sup(A)$. If $\varepsilon > 0$ then there is some $x \in A$ so that $\alpha - \varepsilon < x \leq \alpha$.

16. Prove: Let $\alpha = \sup(A)$. If $\alpha \notin A$ then A is infinite.

17. Solve expressions like: $x^4 = 1$, $x^3 = 2$ and $x^4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

4 Sequences

18. Show a given sequence is convergent.

    (a) $\lim_{n\to\infty} \frac{n+2}{n^2+4} = 0$

    (b) $\lim_{n\to\infty} \frac{n+2}{n+4} = 1$

    (c) $\lim_{n\to\infty} \frac{3n+2}{2n+\sqrt{n}+4} = \frac{3}{2}$

    (d) $\lim_{n\to\infty} \frac{n+2}{3n+\sin n+4} = \frac{1}{3}$

19. Prove if a sequence is convergent then it is bounded.

20. Know how to prove a property like: If (a_n) and (b_n) are convergent then $(a_n + b_n)$ is convergent. We learned four properties like this.

21. State and prove the Monotone convergence Theorem.

22. Use the MCT to prove convergence for a recursively defined sequence.

23. For the following questions use the Monotone Convergence Theorem and the sequence

$$a_1 = 1 \text{ and } a_{n+1} = 2 - \frac{1}{a_n + 1}.$$

- Write the first four terms of the series.
- Show (a_n) is monotone.
- Show (a_n) is bounded.
- Prove (a_n) is convergent.
- What is the limit of (a_n).

5 Limits of Functions

24. Compute the following limits and use the $\varepsilon - \delta$ definition to prove it.

    (a) $\lim_{x \to 3} x^2 + 2x$

    (b) $\lim_{x \to c} x^2 + 2x$

    (c) $\lim_{x \to 9} \sqrt{x} = 3$

25. Use the $\varepsilon - \delta$ definition to prove

    (a) If $\lim_{x \to c} f(x) = F$ and $k \in \mathbb{R}$ then ${}_{x \to c} kf(x) = kF$

    (b) If $\lim_{x \to c} f(x) = F$ and $\lim_{x \to c} g(x) = G$ then $\lim_{x \to c} f(x) + g(x) = F + G$

26. Use the fact that

$$\lim_{a \to 0} \frac{\sin(a)}{a} = 1$$

to solve the following (do not use $\varepsilon - \delta$):

    (a) $\lim_{x \to 0} \frac{1 - \cos(x)}{x}$

    (b) $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$

    (c) $\lim_{x \to 0} \frac{\sin^2(x)}{x^2}$

    (d) $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$. Hint use Problem **??**.