Math 6250: Final Exam Review

1 Limits of Sequences

1. Prove the following
   - $\lim_{n\to\infty} \frac{1}{3n} = 0$
   - $\lim_{n\to\infty} \frac{3n^2+1}{4n^2-7} = \frac{3}{4}$
2. Assume $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $k \in \mathbb{R}$. Prove the following.
   - $\lim_{n\to\infty} k = k$
   - $\lim_{n\to\infty} ka_n = ka$
   - $\lim_{n\to\infty} a_n + b_n = a + b$
   - $\lim_{n\to\infty} a_n b_n = ab$
3. Prove if (a_n) converges then (a_n) is bounded.
4. For the following questions use the Monotone Convergence Theorem and the sequence
   $$a_1 = 1 \text{ and } a_{n+1} = \sqrt{a_n + 1}.$$
   - Write the first four terms of the series.
   - Show (a_n) is monotone.
   - Show (a_n) is bounded.
   - Prove (a_n) is convergent.
   - What is the limit of (a_n).
5. For the following questions use the Monotone Convergence Theorem and the sequence
   $$a_1 = 4 \text{ and } a_{n+1} = \frac{5}{6 - a_n}.$$
   - Write the first four terms of the series.
   - Show (a_n) is monotone.
   - Show (a_n) is bounded.
   - Prove (a_n) is convergent.
   - What is the limit of (a_n).

2 Limits of Functions

6. Prove the following (use $\varepsilon - \delta$ definition).
   (a) $\lim_{x\to -2} 3x - 1 = -7$
   (b) $\lim_{x\to -2} x^2 = 4$
   (c) $\lim_{x\to 4} \frac{1}{x} = \frac{1}{4}$
   (d) $\lim_{x\to 9} \sqrt{x} = 3$

7. Prove the following
   (a) $f(x) = x^2$ is continuous at $x = -2$.
   (b) $f(x) = x^2$ is continuous.
   (c) $f(x) = \sqrt{x}$ is continuous at $x = 0$.
   (d) $f(x) = \sqrt{x}$ is continuous at $x = 4$.
   (e) $f(x) = \sqrt{x}$ is continuous.
   (f) $f(x) = \frac{1}{x}$ is continuous at $x = 3$.
   (g) $f(x) = \frac{1}{x}$ is continuous.

8. Use the $\varepsilon - \delta$ definition to prove
   (a) If $\lim_{x\to c} f(x) = F$ and $k \in \mathbb{R}$ then ${}_{x\to c} kf(x) = kF$
   (b) If $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ then $\lim_{x\to c} f(x) + g(x) = F + G$
   (c) If $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ then $\lim_{x\to c} f(x)g(x) = FG$

9. Use the fact that
$$\lim_{a\to 0} \frac{\sin(a)}{a} = 1$$
   to solve the following (do not use $\varepsilon - \delta$):
   (a) $\lim_{x\to 0} \frac{1-\cos(x)}{x}$
   (b) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$
   (d) $\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$. Hint use The following identity.
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
   or
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

3 Continuity

10. Show the following functions are continuus (or not).

    (a) $f(x) = x^3$ at $x = c$
    (b) $f(x) = \frac{1}{x}$ at $x = c$
    (c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$
    (d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$
    (e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$
    (f) $f(x) = \frac{x^4+x^2}{|x|}$ at $x = 0$
    (g) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
    (h) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
    (i) $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
    (j) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$
    (k) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

4 Derivatives

11. Prove if f is differentiable at $x = c$ then f is continuous at $x = c$.

12. Prove if f is differentiable at $x = c$ and g is differentiable at $x = c$ then fg is continuous at $x = c$.

13. From the definition compute the derivatives for the following functions.

    (a) $f(x) = x^3$ at $x = c$
    (b) $f(x) = \frac{1}{x}$ at $x = c$
    (c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$
    (d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$

(f) $f(x) = \frac{x^4+x^2}{|x|}$ at $x = 0$

(g) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(h) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(i) $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(j) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(k) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

5 Taylor and other series

14. Be able to prove the harmonic series diverges.

15. Be able to compute Taylor polynomials.

16. be able to replicate the proof of the Basel problem (in pieces).

6 Complex Analysis and Fourier Series

17. For $\omega^4 = 1$ solve for ω.

18. For $\omega^3 = 1$ solve for ω.

19. For $\omega^8 = 1$ solve for ω.

20. Graph the image of a set of points in $\mathbb{C}$.

    (a) For $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ graph the image of the following sets

    - $A = \{i\}$
    - $B = \{e^{i\pi//4}, e^{i\pi//2}\}$
    - $C = \{e^{i\pi//4}\}$
    - $D = \{z : |z| < 1\}$
    - $E = \{z : \pi//4 < \theta < \pi//4\}$

(b) For $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \sqrt{z}$ graph the image of the following sets

- $A = \{i\}$
- $B = \{e^{i\pi//4}, e^{i\pi//2}\}$
- $C = \{e^{i\pi//4}\}$
- $D = \{z : |z| < 1\}$
- $E = \{z : \pi//4 < \theta < \pi//4\}$

(c) For $f : \mathbb{C} \to \mathbb{C}$, $f(z) = 3z - 1$ graph the image of the following sets

- $A = \{i\}$
- $B = \{e^{i\pi//4}, e^{i\pi//2}\}$
- $C = \{e^{i\pi//4}\}$
- $D = \{z : |z| < 1\}$
- $E = \{z : \pi//4 < \theta < \pi//4\}$

21. Find the discrete Fourier representation (symmetric decompoitsition) of the following functions for the given n. Recall

$$f_j = \frac{1}{n} \sum_{k=0}^{n-1} f(\omega^k z)\omega^{-jk}.$$

- $f(z) = 3z - 1$, $n = 4$
- $f(z) = \frac{1}{z}$, $n = 4$
- $f(z) = 3z - 1$, $n = 2$
- $f(z) = \sqrt{z}$, $n = 3$