C is

(16) $\oint_C xy\,dx$ $\quad$ $(0,0)$ to $(0,2)$ to $(1,2)$ to $(0,0)$

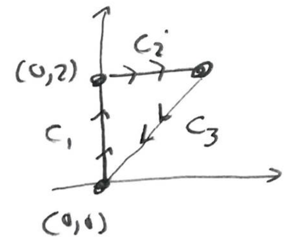
SOL'N $\boxed{C_1}$

$x = 0 + 0t = 0$

$y = 0 + 2t = 2t$

INITIAL POINT $(0,0)$

FINAL − INITIAL $= \langle 0,2\rangle - \langle 0,0\rangle = \langle 0,2\rangle$



$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$0 \le t \le 1$

$\vec{r}(t) = \langle 0, 2t\rangle$

$x = 0$ $\quad$ $y = 2t$

$dx = 0$

$$\int_{C_1} xy\,dx = \int 0(2t)0 = 0$$

$\boxed{C_2}$

$x = 0 + t$ $\quad$ $0 \le t \le 1$

$y = 2 + 0t$

$\vec{r}(t) = \langle t, 2\rangle$ So $dx = 1\,dt$

$$\int_{C_2} xy\,dx = \int_0^1 (t)(2)\,dt = t^2\Big|_0^1 = 1$$

$\boxed{C_3}$

$x = 1 - t$ $\quad$ $0 \le t \le 1$

$y = 2 - 2t$

$dx = -dt$

$$\int_{C_3} xy\,dx = -\int_0^1 (1-t)(2-2t)\,dt = -\int (2 - 4t + 2t^2)\,dt$$

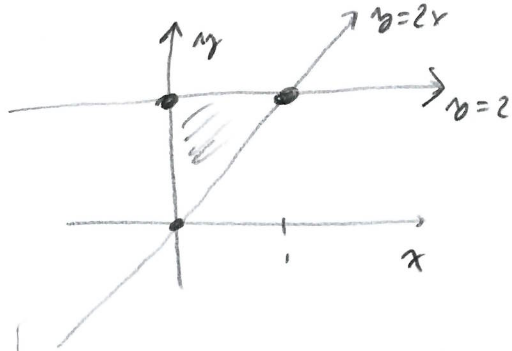
$$= -\left[2t - 2t^2 + \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}$$

So the answer is

$$\oint_C xy\,dx = \int_{C_1} xy\,dx + \int_{C_2} xy\,dx + \int_{C_3} xy\,dx = 0 + 1 - \frac{2}{3} = 1/3$$

(16.) another way

$$\oint_C xy\,dx$$

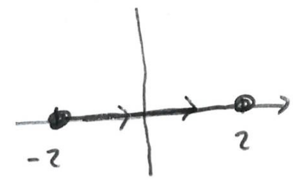


$$\oint_C xy\,dx = \oint_C xy\,dx + 0\,dy \overset{GT}{=} -\iint \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(xy)\right] dA$$

$$= -\iint (-x)\,dA = \int_0^1\int_{2x}^2 x\,dy\,dx = \int_0^1 \left[xy\right]_{2x}^2 dx = \int_0^1 \left[2x - 2x^2\right] dx$$

$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \left[1 - \frac{2}{3}\right] - \left[0\right] = 1/3$$

22 a) $\int_C \langle y^2+6x, 2xy\rangle \cdot d\vec{r}$ $\quad$ $(-2,0)$ to $(2,0)$



Sol'n

Is $\vec{F}$ conservative? $\vec{F} = \langle y^2+6x, 2xy\rangle$

$f(x,y) = y^2+6x$ $\quad$ $g(x,y) = 2xy$

$f_y(x,y) = 2y$ $\quad$ $g_x(x,y) = 2y$

$f_y = g_x \Rightarrow \vec{F}$ is conservative

Find p

$p = \int (y^2+6xy)\,dx = y^2x + 3x^2y + C(y)$

$p = \int (2xy)\,dy = xy^2 + C(x)$

So $p(x,y) = y^2x + 3x^2y$

So $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\vec{p}\cdot d\vec{r} = p(x,y)\Big|_{(-2,0)}^{(2,0)} = y^2x + 3x^2y\Big|_{(-2,0)}^{(2,0)}$

$= [0+0] - [0+0] = 0.$

$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}$

(23.) $\oint \langle x, -y \rangle \cdot d\vec{r}$

$$\overset{GT}{=} -\iint \left[\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x)\right] dA = 0.$$

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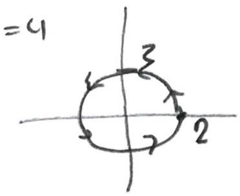
(24) $\oint \langle e^{x^3} - xy, e^{y^3} - y \rangle \cdot d\vec{r}$

$$\overset{GT}{=} -\iint \left[\frac{\partial}{\partial x}(e^{y^3} - y) - \frac{\partial}{\partial y}(e^{x^3} - xy)\right] dA$$

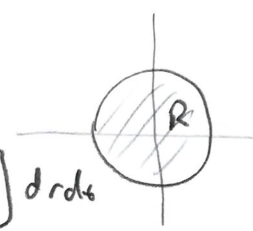
$$= -\iint \left[\ 0 - (-x)\ \right] dA = -\int_0^1 \int_{2x}^{2} x \, dy \, dx = \cdots = -\frac{1}{3}$$

(25.) $\oint \langle \cos(x^2) + y, \cos(y^2) + xy \rangle \cdot d\vec{r}$ $\quad C \quad x^2 + y^2 = 4$



$$\overset{GT}{=} \iint \left[\frac{\partial}{\partial x}\left(\cos(y^2) + xy\right) - \frac{\partial}{\partial y}\left(\cos(x^2) + y\right)\right] dA$$

$$= \iint_R (y - 1) \, dA$$



$$= \int_0^{2\pi}\int_0^2 [r\sin\theta - 1]\, r \, dr \, d\theta = \int_0^{2\pi}\int [r^2 \sin\theta - r] \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\sin\theta - \frac{r^2}{2}\right]_0^2 d\theta = \int_0^{2\pi} \left[\frac{8}{3}\sin\theta - 2 - (0)\right] d\theta = -\frac{8}{3}\cos\theta - 2\theta \Big|_0^{2\pi}$$

$$= \left(-\frac{8}{3}\cos(2\pi) - 2(2\pi)\right) - \left(-\frac{8}{3}\cos(0) - 2(0)\right) = \boxed{-4\pi}$$

(27 a) $\iint z\,dS$ $\qquad x^2+y^2+z^2=4 \qquad z\geq 0$

Sol'n

$x = 2\cos(u)\ \sin(v)$	$0\leq u\leq 2\pi$
$y = 2\sin(u)\ \sin(v)$	$0\leq v\leq \pi/2$
$z = 2\cos(v)$	

$$\vec{r}(u,v) = \langle 2\cos(u)\sin(v),\ 2\sin(u)\sin(v),\ 2\cos(v)\rangle$$

$$\vec{r}_u = \langle -2\sin(u)\sin(v),\ 2\cos(u)\sin(v),\ 0\rangle$$

$$\vec{r}_v = \langle 2\cos(u)\cos(v),\ 2\sin(u)\cos(v),\ -2\sin(v)\rangle$$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin(u)\sin(v) & 2\cos(u)\sin(v) & 0 \\ 2\cos(u)\cos(v) & 2\sin(u)\cos(v) & -2\sin(v)\end{vmatrix}$$

$$= \hat{i}\left(-4\cos(u)\sin^2(v) - 0\right) - \hat{j}\left(4\sin(u)\sin^2(v) - 0\right) + \hat{k}\left(-4\sin^2 u\ \sin(v)\cos(v) - 4\cos^2(u)\sin(v)\cos(v)\right)$$

$$= -4\left\langle \cos(u)\sin^2 v,\ \sin(u)\sin^2(v),\ \underbrace{\left[\sin^2(u)+\cos^2(u)\right]}_{=1}\cdot\sin(v)\cos(v)\right\rangle$$

$$= -4\left\langle \cos(u)\sin^2 v,\ \sin(u)\sin^2 v,\ \sin(v)\cos(v)\right\rangle$$

$$\|\vec{r}_u\times\vec{r}_v\| = 4\sqrt{\cos^2 u\sin^4 v + \sin^2 u\sin^4 v + \sin^2 v\cos^2 v}$$

$$= 4\sqrt{\underbrace{(\cos^2 u + \sin^2 u)}_{=1}\sin^4(v) + \sin^2 v\cos^2 v}$$

$$= 4\sqrt{\sin^2(v)\cdot\underbrace{\left[\sin^2 v + \cos^2 v\right]}_{=1}}$$

$$= 4\sin(v),$$

(27a) CONTINUED

$dS = \|\vec{r}_u \times \vec{r}_v\| \, du\,dv = 4 \sin(v) \, du\,dv$

$$\iint z \, dS = \int_0^{2\pi}\int_0^{\pi/2} (\ 2\cos(v)\ )\, 4\sin(v)\, du\,dv$$

$$= 8 \int_0^{2\pi} \left(\tfrac{1}{2}\sin^2(v)\right)\Big|_0^{\pi/2} du = 8\left[\tfrac{1}{2}\underbrace{\sin^2(\tfrac{\pi}{2})}_{=1} - \tfrac{1}{2}\sin^2(0)\right] \int du$$

$$= 4 \cdot u\Big|_0^{\pi/2} = \boxed{2\pi}$$

(27b) Same parametrization & same $\|\vec{r}_u \times \vec{r}_v\|$ So

$dS = 4 \sin v \, du\,dv$. Compute

$$\iint (x - 2y)\, dS = \iint \left(2\cos u \sin(v) - 2\sin(u)\sin(v)\right) 4\sin v \, dv\,du$$

$$= \iint \left(2\cos u - 2\sin(u)\right) 4\sin^2 v \, dv\,du = 2\int_0^{2\pi} (\cos u - \sin(u))\,du \cdot \int_0^{\pi/2} 4\sin^2(v)\,dv$$

$$= \ldots = 0$$

(28.) $\vec{F} = \langle x, 2y, -3z \rangle$
(a)

$15x - 12y + 3z = 6$

So $z = 2 - 5x + 4y$

$0 \le x \le 1$ AND $0 \le y \le 1$

$\vec{r}(u,v) = \langle u, v, 2 - 5u + 4v \rangle$

$0 \le u \le 1$
$0 \le v \le 1$

$\vec{r}_u = \langle 1, 0, -5 \rangle$

$\vec{r}_v = \langle 0, 1, 4 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 5, -4, 1 \rangle$

So $\iint_S \vec{F} \cdot d\vec{S} = \iint \langle x, 2y, -3z \rangle \cdot d\vec{S}$

$= \int_0^1 \int_0^1 \langle u, 2v, -3(2 - 5u + 4v) \rangle \cdot \langle 5, -4, 1 \rangle \, du\, dv$

$= \int_0^1 \int_0^1 \left[5u + -8v + -6 + 15u - 12v \right] du\, dv$

$= \int_0^1 \int_0^1 \left[20u - 20v - 6 \right] du\, dv = \int_0^1 \left[10u^2 - 20vu - 6u \right]_0^1 dv$

$= \int_0^1 \left[(10 - 20v - 6) - 0 \right] dv = \int (4 - 20v)\, dv = 4v - 10v^2 \Big|_0^1$

$= \boxed{-6}$

(28b) $\vec{F} = \langle x, y, 0 \rangle$ $\quad S: \; z = \sqrt{1 - x^2 - y^2}$

$x = u \qquad -1 \le u \le 1$

$y = v \qquad -\sqrt{1-u^2} \le v \le +\sqrt{1-u^2}$

$z = \sqrt{1 - u^2 - v^2}$

$\vec{r}(u,v) = \langle u, v, (1-u^2-v^2)^{1/2} \rangle$

$r_u(u,v) = \langle 1, 0, -u(1-u^2-v^2)^{-1/2} \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, -v(1-u^2-v^2)^{-1/2} \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -u(1-u^2-v^2)^{-1/2} \\ 0 & 1 & -v(1-u^2-v^2)^{-1/2} \end{vmatrix}$$

$$= \hat{\imath}\left(0 + u(1-u^2-v^2)^{-1/2}\right) - \hat{\jmath}\left(-v(1-u^2-v^2)^{-1/2} - 0\right) + \hat{k}(1-0)$$

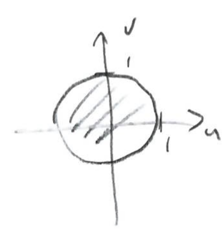
$$= \langle u(1-u^2-v^2)^{-1/2}, v(1-u^2-v^2)^{-1/2}, 1 \rangle.$$

$$\iint \vec{F} \cdot d\vec{S} = \iint \langle x, y, 0 \rangle \cdot d\vec{S}$$

$$= \iint \langle u, v, 0 \rangle \cdot \langle u(1-u^2-v^2)^{-1/2}, v(1-u^2-v^2)^{-1/2}, 1 \rangle \, du\,dv$$

$$= \iint \left(u^2(1-u^2-v^2)^{-1/2} + v^2(1-u^2-v^2)^{-1/2} + 0\right) du\,dv$$

$$= \iint_R \frac{u^2+v^2}{\sqrt{1-u^2-v^2}} \, du\,dv$$



(28b CONTINUED)

$$= \int_0^{2\pi}\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, r\, dr\, d\theta = \int_0^1 \frac{r^3}{(1-r^2)^{1/2}}\, dr \cdot \underbrace{\left(\int_0^{2\pi} 1\, d\theta\right)}_{=\pi}$$

$$= 2\pi \int \frac{r^3}{\sqrt{1-r^2}}\, dr \qquad w = 1-r^2 \quad \text{so } r^2 = 1-w \qquad dw = -2r\,dr$$

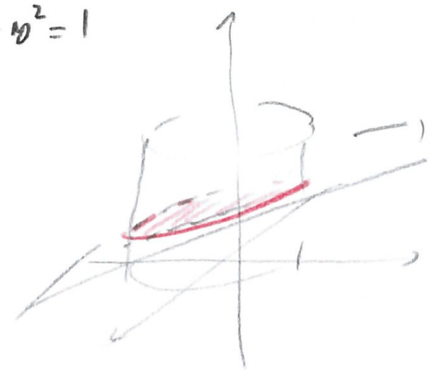
$$= 2\pi \int \frac{r^2}{\sqrt{1-r^2}}\, \underbrace{r\,dr}_{-\frac{1}{2}dw} = 2\pi \int \frac{1-w}{w^{1/2}}\left(-\tfrac{1}{2}\,dw\right) = -\pi \int \left(w^{-1/2} - w^{1/2}\right) dw$$

$$= -\pi\left[\frac{w^{1/2}}{1/2} + \frac{w^{3/2}}{3/2}\right] = -\pi\left[2(1-r^2)^{1/2} + \frac{2}{3}(1-r^2)^{3/2}\right]_0^1$$

$$= -\pi\left[(0+0) - \left(2+\frac{2}{3}\right)\right] = \boxed{\frac{8\pi}{3}}$$

(28 c) $\vec{F} = \langle x^2, y^2, z^2 \rangle$

S : $z = y + 1$ inside $x^2 + y^2 = 1$

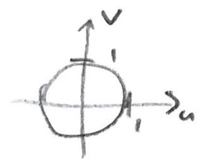


0. PARAMETRIZATION

$$\vec{r}(u,v) = \langle u, v, v+1 \rangle$$

$$-1 \le u \le +1$$

$$-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$$



1. COMPUTE $d\vec{S}$

$$\vec{r}_u = \langle 1, 0, 0 \rangle$$

$$r_v = \langle 0, 1, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0, -1, 1 \rangle$$

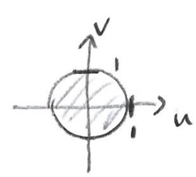
2. Set up INTEGRAL

$$\iint \vec{F} \cdot d\vec{S} = \iint \langle x^2, y^2, z^2 \rangle \cdot d\vec{S}$$

$$= \iint_R \langle u^2, v^2, (v+1)^2 \rangle \cdot \langle 0, -1, 1 \rangle \cdot du\,dv$$

$$= \iint_R \left[0 + -v^2 + (v+1)^2 \right] du\,dv = \iint \left[-v^2 + v^2 + 2v + 1 \right] du\,dv$$

$$= \iint_R \left[2v + 1 \right] du\,dv$$



use polar

$0 \le r \le 1$ $\quad u = r\cos\theta$

$0 \le \theta \le 2\pi$ $\quad v = r\sin\theta$

$$= \int_0^{2\pi} \int_0^1 \left[2r\sin\theta + r \right] r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[2\sin\theta + 1 \right] d\theta \cdot \int_0^1 r^2\,dr = \left[-2\cos\theta + \theta \right]_0^{2\pi} \cdot \frac{r^3}{3}\Big|_0^1$$

$$= (2\pi) \cdot \frac{1}{3} = \boxed{\frac{2}{3}\pi}$$