Math 3160 - Final Review

Be certain to **know** the Test 1 Review, Test 2 Review and . . .

1 Change of Basis Matrix

1. Let $B = \{(1,0),(0,1)\}$, $B_1 = \{(-1,1),(2,3)\}$ and $B_2 = \{(1,-1),(1,1)\}$.

   (a) Find the change of basis matrices for $P_{B_1 \to B_2}$ and $P_{B_1 \to B_2}$.

   (b) Find the coordinates of the point $(4,6)$ (given in the standard basis) relative to the bases B_1 and B_2.

   (c) Find the change of basis matrices for $P_{B \to B_2}$ and $P_{B_2 \to B}$.

   (d) Find the coordinates of the point $(2,-4)$ (given in the standard basis) relative to the bases B and B_2. Graph this point the two separate coordinate axes B and B_2.

2 Column Space & Null space

2. Let $A = \begin{bmatrix} -1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & -1 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{bmatrix}$.

   (a) Find a basis for the Column Space of A, COL(A).

   (b) Compute the dimension of COL(A).

   (c) Find a basis for the null space of A, NULL(A).

   (d) Compute the dimension of NULL(A).

3. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the formula $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+y \\ y+z \\ x-z \end{bmatrix}$.

   (a) Find the matrix, A, to represent the linear transformation T.

   (b) Compute the basis for the Range of T, which is the Column Space of A.

   (c) Find a basis for the null space of A, NULL(A).

(d) Compute the dimension of COL(A) and NULL(A). The dimension of the range of T is called the rank of T and the dimension of the null space is called the nullity.

(e) What is the dimension of the domain of T and the codomain of T? Again, compare Rank, Nullity and the dimension of the Domain. Do you see a relation?

3 Diagonalizations

4. Find the eigenvalues and eigenvetors for the following matrices

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}.$$

5. Find the eigenvalues and eigenvetors for the following matrices (may have complex numbers in answer).

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix}.$$

6. Can the following matricses be diagonalized? State why? If it can be diagonalized, do it.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}.$$

4 Recursively Defined Sequences

7. Let a sequence be defined by the following recursive formula

$$a_1 = 3, a_2 = 0 \text{ and } a_{n+2} = 3a_{n+1} - 2a_n$$

(a) Compute the first five terms of the sequence.

(b) Find a matrix A for the sequence as we did in class.

(c) Diagonalize A. That is find D and P so that $A = PDP^{-1}$.

(d) Compute $A^n \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ using Problem 7c.

8. Let a sequence be defined by the following recursive formula

$$a_1 = 4, a_2 = 1 \text{ and } a_{n+2} = a_{n+1} - 6a_n$$

(a) Compute the first five terms of the sequence.

(b) Find a matrix A for the sequence as we did in class.

(c) Diagonalize A. That is find D and P so that $A = PDP^{-1}$.

(d) Compute $A^n \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ using Problem 8c.

5 Cryptography - Hill Cipher

9. Which of the following can be used to encipher with the Hill cipher (as we did in class mod 26)?

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 25 & 5 \\ 3 & 11 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

10. Define the follwing matrix: $E = \begin{bmatrix} 5 & 1 \\ 3 & 6 \end{bmatrix}$

(a) For the plaintext message “WOLFMAN”, find the two letter block representation using the alphabet: $A = 00, B = 01, C = 02, D = 03, \ldots$

(b) Use the matrix E to encipher the plaintmessage.

(c) Find the deciphering matrix D.

(d) Decipher the ciphertext: JBCMVPTKGD.

6 Markov Chain - Won't be on Final Exam

11. For the following defined matrices which are stochastic. $A = \begin{bmatrix} .1 & .9 \\ .1 & .9 \end{bmatrix}$,

$B = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0.25 & -1.0 \\ 0.75 & 2.0 \end{bmatrix}$ and $D = \begin{bmatrix} .1 & .2 & .3 & .4 \\ .0 & .4 & .1 & .5 \\ .0 & .4 & .6 & 0 \\ .9 & 0 & 0 & .1 \end{bmatrix}$

12. For the following defined markov processes which are regular.
$A = \begin{bmatrix} .1 & .9 \\ .1 & .9 \end{bmatrix}$, $B = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0.25 & 1.0 \\ 0.75 & 0.0 \end{bmatrix}$

13. Find the steady states for the following processes.

    (a) For the transition matrix $A = \begin{bmatrix} .1 & .9 \\ .9 & .1 \end{bmatrix}$. You may be able to guess the answer for this one (but do the work anyway).

    (b) A bike rental company has three locations. A renter can pick up a bike at any location and drop that bike off at any location.

    - A bike picked up at location A has 20% probability of being dropped off at location B and has a 10% probability of being dropped off at location C.
    - A bike picked up at location B has 30% probability of being dropped off at location A and has a 0% probability of being dropped off at location C.
    - A bike picked up at location C has 10% probability of being dropped off at location B and has a 40% probability of being dropped off at location A.