(c) Compute $C^T C$. What kind of matrix is $C^T C$?

4. Solve the following equations for X assuming any matrix has an inverse. Let A, B, C, X be $n \times n$ matrices and let $\mathbf{u}$ be an $n \times 1$ vector.

   (a) $AX = BX - A$

   (b) $AX = 2X - A$

   (c) $A\mathbf{u} = 2\mathbf{u} + B$

5. Solve the following using Cramer's Rule.

$$\begin{cases} x & -2y & = 0 \\ 5x & -2y & = 0 \end{cases}$$

$$\begin{cases} x_1 & -2x_2 & & = 0 \\ x_1 & & +x_3 & = 0 \\ & x_2 & +3x_3 & = 0 \end{cases}$$

3 Linear Transformations

6. For the following transformation. What is A? What is the dimension of the domain? What is the dimmension of the codomain?

$$\begin{aligned} w_1 &= x_1 + x_3 \\ w_2 &= 3x_2 - x_3 \end{aligned}$$

7. For the following transformation (T), what is A? What is the dimension of the domain? What is the dimmension of the codomain? Compute the value of $T(e_2)$

$$\begin{aligned} w_1 &= x_1 + x_3 \\ w_2 &= 3x_2 - x_3 \end{aligned}$$

8. For the following matrix as a linear transformation, T, what is the dimension of the domain? What is the dimmension of the codomain? Compute the value of $T(e_1)$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

9. What transformation, $T : \mathbb{R}^2 \to \mathbb{R}^3$ has the following properties

$$T(e_1) = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, T(e_2) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}.$$

Math 3160 - Test 1 Review

1 Systems of Linear Equations

1. Solve the following systems of linear equations using row reduction.

(a) $\begin{cases} x_1 & -2x_2 & & & -6x_5 & = 0 \\ & x_2 & +x_3 & +6x_4 & & = 5 \\ & 2x_2 & & +6x_4 & +x_5 & = 4 \\ & x_2 & -x_3 & & +x_5 & = -1 \end{cases}$

(b) $\begin{cases} 2x_1 & -2x_2 & +4x_3 & = 2 \\ & & x_3 & = 0 \\ x_1 & +x_2 & +2x_3 & = 0 \end{cases}$

(c) $\begin{cases} 2x_1 & -2x_2 & +4x_3 & = 2 \\ -x_1 & -x_2 & +3x_3 & = 2 \\ x_1 & -3x_2 & +7x_3 & = 2 \end{cases}$

2. Solve the following systems of linear equations by setting up problem as a matrix problem and by finding an inverse matrix.

(a) $\begin{cases} 2x_1 & -2x_2 & +4x_3 & = 2 \\ & -x_2 & +3x_3 & = 2 \\ & -3x_2 & +7x_3 & = 2 \end{cases}$

(b) $\begin{cases} 2x_1 & -2y & = 2 \\ -x_1 & -3y & = 2 \end{cases}$

2 Matrices, Determinants, Cramer's Rule

3. Let $A = \begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & 1 \\ 0 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 & 0 & 0 \\ 2 & -2 & 1 & 2 \\ 2 & -2 & 0 & 3 \\ 2 & -2 & 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the determinant of the matices A, B, C and D

(b) Compute D^3 and D^{-1}.

I will skip the work. You must show your work!

1.

(a) $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & -6 & 0 \\ 0 & 1 & 1 & 6 & 0 & 5 \\ 0 & 2 & 0 & 6 & 1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 6 & -5 & 0 \\ 0 & 1 & 0 & 3 & -1/2 & 0 \\ 0 & 0 & 1 & 3 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

I skipped many steps

The last row is $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1$

$0 = 1$

Bad. No solutions, write $S = \{ \}$.

(b.) $\left[\begin{array}{ccc|c} 2 & -2 & 4 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

So $x_1 = 1/2$, $x_2 = -1/2$, AND $x_3 = 0$. So $S = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} \right\}$.

(c) $\left[\begin{array}{ccc|c} 2 & -2 & 4 & 2 \\ -1 & -1 & 3 & 2 \\ 1 & -3 & 7 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$0 = 1 \Rightarrow S = \{ \}$.

Also be prepared for an ∞ly many solutions

Like $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$

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2 (a) Find Determinants

$|A| = \det \begin{bmatrix} 2 & -2 & 0 \\ -1 & 4 & 1 \\ 0 & 4 & 1 \end{bmatrix} = 0 \cdot \begin{vmatrix} -1 & 4 \\ 0 & 4 \end{vmatrix} - 1 \begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix} + 1 \begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix}$

$= 0 - 1(8-0) + 1(8-2) = -8+6 = -2$

$|B| = 0$

---

2 (a) CONTINUED

$$|C| = \begin{vmatrix} 2 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 0 \cdot |?| - 0|?| + 0|?| - 0|?| + 1 \begin{vmatrix} 2 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$= 0 - 0 + 0 - 0 + 1 \begin{vmatrix} 2 & -2 & 0 & -2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$= 0 + 1 \cdot \left[2 \cdot \begin{vmatrix} -2 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & -2 \end{vmatrix} - 0 \cdot |?| + 0 \cdot |?| - 0 |?| \right]$$

... steps missing

$$= 24$$

$$|D| = 8$$

(b) $$D^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\cdots \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(c) $$C^T C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -2 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 & -4 & 0 \\ -4 & 8 & 0 & 8 & 0 \\ 0 & 0 & 9 & -6 & 0 \\ -4 & 8 & -6 & 16 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A Symmetric matrix!

6. Let T be the Linear transform defined below

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ 2y - z \\ 3x + z \end{pmatrix}$$

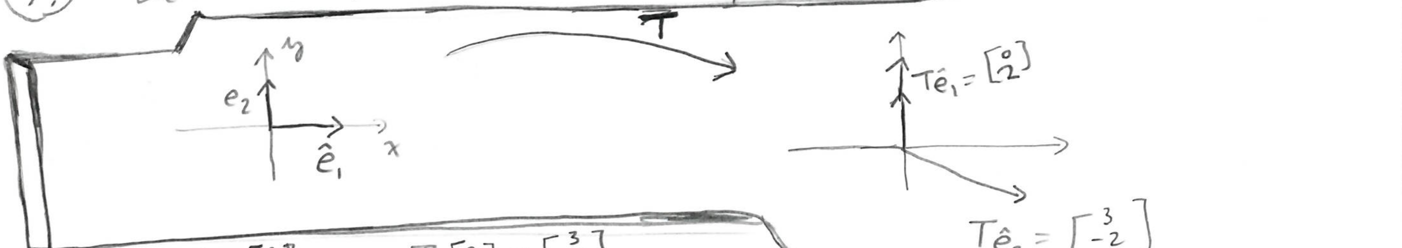
(a.) $T: \mathbb{R}^n \to \mathbb{R}^m$ - what is n? AND m?

(b.) Write the matrix A so that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

---

7. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by



So $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ AND $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(a) FIND A so that $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) FIND A^{-1}.

(c) Compute $A^{-1} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ AND $A^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

(4.) (a) $Ax = Bx - A$

$Ax - Bx = -A$

$(A - B)x = -A$

$(A-B)^{-1}(A-B)\,x = (A-B)^{-1}(-A)$

$x = -(A-B)^{-1}A$

(b) $Ax = 2x - A$

$Ax - 2x = -A$

$Ax - 2Ix = -A$

$(A - 2I)x = -A$

$x = -(A-2I)^{-1}A.$

(c) $A\vec{u} = 2\vec{u} + B$

$A\vec{u} - 2\vec{u} = B$

$A\vec{u} - 2I\vec{u} = B$

$(A - 2I)\vec{u} = B$

$\vec{u} = (A-2I)^{-1}B.$

(5) (a) $x - 2y = 0$

$5x - 2y = 0$

$A = \begin{bmatrix} 1 & -2 \\ 5 & -2 \end{bmatrix}$ $A_1 = \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix}$

$|A| = -2 + 10$
$= 8$

$|A_1| = 0 - 0$
$= 0$

$|A_2| = 0 - 0$
$= 0$

$x_1 = \dfrac{|A_1|}{|A|} = \dfrac{0}{8}$ $x_2 = \dfrac{|A_2|}{|A|} = \dfrac{0}{8} = 0$

So $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

(b) $|A| = \begin{vmatrix} 1 & -2 & 0 \\ \boxed{1 \quad 0 \quad 1} \\ 0 & 1 & 3 \end{vmatrix} = -1 \begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$

$= -1(-6 - 0) + 0 - 1(1 + 2)$

$= 6 - 2 = 4$

$\vec{b}$

$|A_1| = \begin{vmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 0$

So $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{0}{4} = 0$

what are $|A_2|$ AND $|A_3|$?

(6.) $w_1 = x_1 + x_3$
$w_2 = 3x_2 - x_3$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_1 + x_3 \\ 3x_2 - x_3 \end{bmatrix}, \quad T: \mathbb{R}^3 \to \mathbb{R}^2$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \end{bmatrix}$$

$\dim(\mathrm{dom}(T)) = 3$

$\dim(\mathrm{codom}(T)) = 2$

(7.) Same as above.

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} \overset{x_1 = 0}{x_1} + \overset{x_3 = 0}{x_3} \\ \underset{x_2 = 1}{3x_2} - \underset{x_3 = 0}{x_3} \end{bmatrix} = \begin{bmatrix} 0 + 0 \\ 3(1) - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$x_1 = 0$
$x_2 = 1$
$x_3 = 0$

(8.)
$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 + 0x_2 + 0x_3 + 0x_4 \\ 0x_1 + 4x_2 + 0x_3 + 2x_4 \\ x_3 + 3x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ 4x_2 + 2x_4 \\ x_3 + 3x_4 \end{bmatrix}$$

dim

output has 3 entries so

$\dim(\mathrm{codom}(T)) = 3$

input is 4

so $\dim(\mathrm{domain}(T)) = 4$

So $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$$T(e_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(9.) $A = \begin{bmatrix} 2 & -1 \\ 0 & 4 \\ 5 & 0 \end{bmatrix}$ So $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2x - y \\ 4y \\ 5x \end{bmatrix}$.