Math 6250: Final Exam Review

1 Limits of Sequences

1. Prove the following
   - $\lim_{n\to\infty} \frac{1}{3n} = 0$
   - $\lim_{n\to\infty} \frac{3n^2+1}{4n^2-7} = \frac{3}{4}$
2. Assume $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $k \in \mathbb{R}$. Prove the following.
   - $\lim_{n\to\infty} k = k$
   - $\lim_{n\to\infty} ka_n = ka$
   - $\lim_{n\to\infty} a_n + b_n = a + b$
   - $\lim_{n\to\infty} a_n b_n = ab$
3. Prove if (a_n) converges then (a_n) is bounded.
4. For the following questions use the Monotone Convergence Theorem and the sequence

$$a_1 = 1 \text{ and } a_{n+1} = \sqrt{a_n + 1}.$$

   - Write the first four terms of the series.
   - Show (a_n) is monotone.
   - Show (a_n) is bounded.
   - Prove (a_n) is convergent.
   - What is the limit of (a_n).

2 Limits of Functions

5. Prove the following (use $\varepsilon - \delta$ definition).
   (a) $\lim_{x\to -2} 3x - 1 = -7$
   (b) $\lim_{x\to -2} x^2 = 4$
   (c) $\lim_{x\to 4} \frac{1}{x} = \frac{1}{4}$

(d) $\lim_{x\to 9}\sqrt{x} = 3$

6. Prove the following

   (a) $f(x) = x^2$ is continuous at $x = -2$.
   (b) $f(x) = x^2$ is continuous.
   (c) $f(x) = \sqrt{x}$ is continuous at $x = 0$.
   (d) $f(x) = \sqrt{x}$ is continuous at $x = 4$.
   (e) $f(x) = \sqrt{x}$ is continuous.
   (f) $f(x) = \frac{1}{x}$ is continuous at $x = 3$.
   (g) $f(x) = \frac{1}{x}$ is continuous.

7. Use the $\varepsilon - \delta$ definition to prove

   (a) If $\lim_{x\to c} f(x) = F$ and $k \in \mathbb{R}$ then ${}_{x\to c}kf(x) = kF$
   (b) If $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ then $\lim_{x\to c} f(x) + g(x) = F + G$
   (c) If $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ then $\lim_{x\to c} f(x)g(x) = FG$

8. Use the fact that
$$\lim_{a\to 0}\frac{\sin(a)}{a} = 1$$
to solve the following (do not use $\varepsilon - \delta$):

   (a) $\lim_{x\to 0}\frac{1-\cos(x)}{x}$
   (b) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$
   (c) $\lim_{x\to 0}\frac{\sin^2(x)}{x^2}$
   (d) $\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$. Hint use The following identity.
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
   or
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

3 Continuity

9. Show the following functions are continuus (or not).

   (a) $f(x) = x^3$ at $x = c$

   (b) $f(x) = \frac{1}{x}$ at $x = c$

   (c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

   (d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

   (e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$

   (f) $f(x) = \frac{x^4+x^2}{|x|}$ at $x = 0$

   (g) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

   (h) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

   (i) $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

   (j) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

   (k) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

4 Derivatives

10. Prove if f is differentiable at $x = c$ then f is continuous at $x = c$.

11. Prove if f is differentiable at $x = c$ and g is differentiable at $x = c$ then fg is continuous at $x = c$.

12. From the definition compute the derivatives for the following functions.

    (a) $f(x) = x^3$ at $x = c$

    (b) $f(x) = \frac{1}{x}$ at $x = c$

    (c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

    (d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$

(f) $f(x) = \frac{x^4+x^2}{|x|}$ at $x = 0$

(g) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

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(j) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(k) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

5 Taylor and other series

13. Be able to prove the harmonic series diverges.

14. Be able to compute Taylor polynomials.

15. be able to replicate the proof of the Basel problem (in pieces).

6 Also Remember

6.1 Injective, Surjective

16. Define $f : (0, \infty) \to (0, 1)$ by $f(x) = \frac{x}{1+x}$.

    (a) Graph this function.

    (b) Prove f is injective.

    (c) Prove f is surjective.

6.2 Cardinality

17. What is the definition A and B have the same cardinality.

18. What is the definition A is countable.

19. List five sets which are countably infinite. List three sets that are uncountable.

20. Show the following sets have the same cardinality

    (a) $\mathbb{N}$ and $\mathbb{Z}$
    (b) $\mathbb{N}$ and $\mathbb{Q}$
    (c) $(0, \infty) \sim (0, 1)$.
    (d) $(2, 4] \sim [3, 7)$.