1 From earlier

1. Let $\mathbf{r}(t) \to \mathbb{R} \to \mathbb{R}^3$ be differentiable so that the $\mathbf{r}$ has a constant norm.
   (a) Prove that the path $\mathbf{r}$ and its derivative are perpandicular.
   (b) Come up with a nonzero example of $\mathbf{r}(t)$.
2. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
   (a) Prove the parallelogram rule
   $$||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2 = 2||\mathbf{v}||^2 + 2||\mathbf{w}||^2$$
   (b) graph some vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ in $\mathbb{R}^2$ to explain the parallelogram rule.

2 Paths

3. Let $\mathbf{r}(t) = \langle t^2 - t, t^3 - 3t^2 + 3\rangle$
   (a) Find the position, velocity and acceleration of the particle at time $t = 2$.
   (b) Graph the position, velocity and acceleration appropriately.
4. Let $\mathbf{r}(t) = \langle \sin(e^{-t}), \cos(e^{-t})\rangle$
   (a) Find the speed function
   $$\frac{ds}{dt} = ||\mathbf{r}'(t)||$$
   (b) Compute the arc length from $t = 0$ to $t = 1$.
   (c) Compute the arc length from $t = 0$ to $t = \infty$.
   (d) What is the graph of $\mathbf{r}(t)$?
5. Let $\mathbf{r}(t) = \langle e^{-t}\sin(t), e^{-t}\cos(t)\rangle$
   (a) Find the speed function
   $$\frac{ds}{dt} = ||\mathbf{r}'(t)||$$
   (b) Compute the arc length from $t = 0$ to $t = 1$.
   (c) Compute the arc length from $t = 0$ to $t = \infty$.
   (d) What is the graph of $\mathbf{r}(t)$?

3 Functions of Several Variables

6. Compute the following limits if they exist. If not show why.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2+1}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$

7. Let $f(x,y) = x^2+y^2$. Consider the points $P(0,2)$ and $Q(1,2)$.

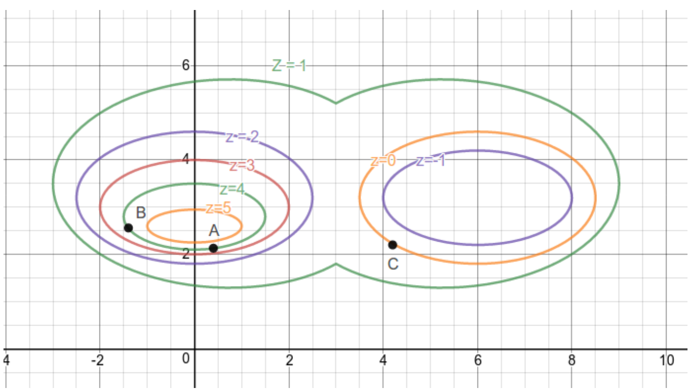
   (a) Graph the contour plot. Include $z = -1, 0, 1, 2, 3, 4$.

   (b) Compute the $\nabla f(x,y)$

   (c) Compute the $\nabla f(P)$ and $\nabla f(Q)$. Also compute their norms.

   (d) Graph $\nabla f(P)$ and $\nabla f(Q)$ with initial points P and Q respectively.

8. Draw the gradient at each point A, B and C.



9. Fill in the blanks.

   (a) The gradient is the direction of __________ increase.

   (b) The gradient is __________ to the contour lines.

   (c) The norm of the gradient is __________.

   (d) A larger norm of one gradient is __________ graphically.

10. Let $f(x,y) = e^{xy^2+2} - xy^3 + 2$. Find the tangent plane to $f(x,y,z)$ at the point $(-2,1)$. Use that plane to estimate $f(-2.1, 0.8)$. Compare to the real value of $f(-2.1, 0.8)$.

11. Let $f(x,y,z) = x^2y - xy^2 + z^3$. Find the tangent plane (actually a hyperplane) to $f(x,y,z)$ at the point $(1,2,3)$

12. Find and classify extremma.

    (a) $f(x, y) = x^2 - xy + y^3$.

    (b) $f(x, y) = x^2 + 2xy - y^4$.

    (c) $f(x, y, z) = x + 3y + z$ subject to $x^2 + y^2 + z^2 = 1$.

    (d) $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to $x + y + z - 3w = 4$.

    (e) $f(x, y, z, w) = x\ln(x) + y\ln(y) + z\ln(z)$ subject to $x + y + z = 1$. In this problem f is called information entropy.

4 Integrals

13. $\iint_R x + y\, dA$ over the region defined by $x + y = 2$ and the coordinate axes.

14. $\iint_R xy\, dA$ over the region defined by $y = x^2$ and the line $y = x + 1$.

15. $\iint_R e^{x^2}\, dA$ over the region defined by $y = -x$, $y = 2x$ and the vertical line $x = 4$.

16. $\iint_R e^{x^2+y^2}\, dA$ over the region defined by the portion of the circle $x^2 + y^2 = 4$ in the third quadrant.

17. $\iint_R \sqrt{\dfrac{\tan^{-1}(y/x)}{x^2 + y^2}}\, dA$ over the region defined by the portion of the circle $x^2 + y^2 = 4$ above the lines $y = -x$ and $y = x$.

18. Find the volume below the paraboloid $z = 12 - x^2 - y^2$ and above the xy-plane.

19. $\iint_R \sin(x - y)\cos(x + y)\, dA$ over the region defined the lines $y = x + 2$, $y = x + 4$, $y = -x$ and $y = -x + 3$. Hint the change of variables is $u = x - y$ and $v = x + y$.

20. $\iint_R \dfrac{x - y}{2x + y}\, dA$ over the region defined the lines $y = x + 2$, $y = x$, $y = -2x + 2$ and $y = -2x + 3$.

21. $\iint_R xy\, dA$ over the region defined the graphs of $xy = 1$, $xy = 3$ and the lines $y = x$ and $y = 3x$. Hint $x = u/v$ and $y = v$.

22. $\iint_R (x - y)e^{x^2-y^2}\, dA$ over the region defined the lines $y = x + 2$, $y = x$, $y = -x$ and $y = -x + 3$.

23. $\iint_R e^{x^2+4y^2}\, dA$ over the region defined by the portion of the ellipse $\frac{x^2}{4}+y^2=1$ in the third quadrant. Hint use the change of variables $x=2v\cos(u)$ and $x=v\sin(u)$. And note I had $\pi \le u \le \frac{3\pi}{2}$

5 Line Integrals

24. $\int_C x\, dx$. Let C be line segment from $(0,1)$ to $(3,2)$.

25. $\int_C xy\, ds$. Let C be line segment from $(0,1)$ to $(3,2)$.

26. $\int_C \langle -x, y\rangle \cdot d\mathbf{r}$. Let C be line segment from $(0,1)$ to $(3,2)$.

27. $\int_C x\, dy$. Let C be line segment from $(0,1)$ to $(3,2)$.

28. $\oint_C xy\, dx$. Let C be outside of the triangle traced from $(0,0)$ to $(0,2)$ to $(1,2)$ and then back to $(0,0)$.

29. $\oint_C \langle -x, y\rangle \cdot d\mathbf{r}$. Let C be outside of the triangle traced from $(0,0)$ to $(0,2)$ to $(1,2)$ and then back to $(0,0)$.

30. $\oint_C \langle 1, xy\rangle \cdot d\mathbf{r}$. Let C be the circle $x^2+y^2=4$ traced counter-clockwise.

31. $\oint_C -x+y ds$. Let C be the circle $x^2+y^2=4$ traced counter-clockwise.

6 Fields and Conservative Fields

32. Graph the following fields
    (a) $\mathbf{F}(x,y)=\langle x,1\rangle$.
    (b) $\mathbf{F}(x,y)=\langle y,x\rangle$.

33. Test if fields are conservative. If it is conservative, find its potential function.
    (a) $\mathbf{F}(x,y)=\langle x,1\rangle$.
    (b) $\mathbf{F}(x,y)=\langle y,x\rangle$.

(c) $\mathbf{F}(x, y) = \langle xy, xy \rangle$.

(d) $\mathbf{F}(x, y) = \langle 4x + y + 2, x + 3y^2 \rangle$.

34. Use the fact that the field is conservative, and the FTVC to solve the following integrals.

(a) $\int_C \langle x^2 + y, x + y^2 \rangle \cdot d\mathbf{r}$ where C is the line segment from $(-1, 0)$ to $(1, 0)$.

(b) $\int_C \langle x^2 + y, x + y^2 \rangle \cdot d\mathbf{r}$ where C is the upper half of the circle $x^2 + y^2 = 1$ starting at $(-1, 0)$ and travelling clockwise to $(1, 0)$.

(c) $\int_C \langle 6e^{3x} + y^2, 2xy - 4\sin(2y) \rangle \cdot d\mathbf{r}$ where C is the piesce of the parabola $y = x^2$ that starts at$(-1, 1)$ and to $(1, 1)$.

7 Green's Theorem

35. $\oint_C \langle x, -y \rangle \cdot d\mathbf{r}$. Let C be outside of the square traced from $(0, 0)$ to $(0, 2)$ to $(1, 2)$ to $(1, 0)$ and then back to $(0, 0)$.

36. $\oint_C \langle e^{x^3} - xy, e^{y^3} - y \rangle \cdot d\mathbf{r}$. Let C be outside of the triangle traced from $(0, 0)$ to $(0, 2)$ to $(1, 2)$ and then back to $(0, 0)$.

37. $\oint_C \langle \cos(x^2) + y, \cos(y^2) + xy \rangle \cdot d\mathbf{r}$. Let C be the circle $x^2 + y^2 = 4$ traced counter-clockwise.

8 Div/Grad/Curl

38. Define

$$f(x, y, z) = x^3 - yz^2 \text{ and } \mathbf{F}(x, y, z) = \langle x^3, yz^2, xy \rangle.$$

Compute the following, if possible, and if not possible state why.

(a) $\text{div}(f(x, y, z))$

(b) $\text{grad}(f(x, y, z))$

(c) $\text{curl}(f(x, y, z))$

(d) $\text{div}(\mathbf{F}(x, y, z))$

(e) $\text{grad}(\mathbf{F}(x, y, z))$

(f) $\text{curl}(\mathbf{F}(x, y, z))$

(g) $\nabla \cdot \mathbf{F}(x, y, z)$

(h) $\nabla \times (\nabla \cdot \mathbf{F}(x, y, z))$

(i) $\nabla \times (\nabla f(x, y, z))$

9 Surface Integrals and Stokes' Theorem

39. Some practice parametizing surfaces. Parametize the following
    (a) The part of the plane $x + 4y - 3z = 12$ above the rectangle $0 \le x \le 4$, $0 \le y \le 4$.
    (b) The part of the paraboloid $z = x^2 + y^2$ above the circle $x^2 + y^2 = 4$.

40. For the following exercises, let S be the hemisphere $x^2 + y^2 + z^2 = 4$, with $z \ge 0$, and evaluate each surface integral, in the counterclockwise direction.
    (a) $\iint_S z dS$
    (b) $\iint_S (x - 2y) dS$
    (c) $\iint_S (x^2 + y^2) z dS$

41. For the following exercises, evaluate

$$\iint \mathbf{F} \cdot \mathbf{N} dS$$

vector field $\mathbf{F}$, where $\mathbf{N}$ is an outward normal vector to surface S.
    (a) $\mathbf{F}(x, y, z) = xi + 2yj - 3zk$, and S is that part of plane $15x - 12y + 3z = 6$ that lies above unit square $0 \le x \le 1$, $0 \le y \le 1$.
    (b) $\mathbf{F}(x, y, z) = xi + yj$, and S is the hemisphere $z = \sqrt{1 - x^2 - y^2}$.x
    (c) $\mathbf{F}(x, y, z) = x^2 i + y^2 j + z^2 k$, and S is the portion of plane $z = y + 1$ that lies inside cylinder $x^2 + y^2 = 1$.

Stokes' Theorem'

$$\int \mathbf{F} \cdot d\mathbf{r} = \iint \text{curl}\mathbf{F} \cdot d\mathbf{S}$$

42. For the following exercises, without using Stokes' theorem, calculate directly both the flux of curl$\mathbf{F} \cdot \mathbf{N}$ over the given surface and the circulation integral around its boundary, assuming all boundaries are oriented clockwise as viewed from above.

43. $\mathbf{F}(x, y, z) = y^2 i + z^2 j + x^2 k$; S is the first-octant portion of plane $x + y + z = 1$.

44. $\mathbf{F}(x, y, z) = zi + xj + yk$; and S is the hemisphere $z = \sqrt{9^2 - x^2 - y^2}$.

45. $\mathbf{F}(x, y, z) = y^2 i + 2xj + 5k$; and S is the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

46. $\mathbf{F}(x, y, z) = zi + 2xj + 3yk$; S is upper hemisphere $z = \sqrt{9 - x^2 - y^2}$.

47. $\mathbf{F}(x, y, z) = (x + 2z)i + (y - x)j + (z - y)k$; S is a triangular region with vertices $(3, 0, 0)$, $(0, 3/2, 0)$, and $(0, 0, 3)$.

48. For the following exercises, use Stokes' theorem to evaluate $\iint_S \text{curl}\mathbf{F} \cdot \mathbf{N} dS$ for the vector fields andtraversed counterclockwise viewed from the origin.

    (a) $\mathbf{F}(x, y, z) = xyi - zj$ and S is the surface of the cube $0 \geq x \geq 1, 0 \geq y \geq 1, 0 \geq z \geq 1$, except for the face where $z = 0$, and using the outward unit normal vector.

    (b) $\mathbf{F}(x, y, z) = xyi + x^2 j + z^2 k$; and S is the intersection of paraboloid $z = x^2 + y^2$ and plane $z = y$, and using the outward normal vector.