Test 3

1. $\boxed{\int \frac{2x^2-7x-14}{(x-5)(x^2+1)}\,dx}$

$$\frac{2x^2-7x-14}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$

$$2x^2-7x-14 = A(x^2+1) + (Bx+C)(x-5)$$

$$= Ax^2 + A + Bx^2 - 5Bx + Cx - 5C$$

$$= (A+B)x^2 + (-5B+C)x + A + -5C$$

$\underline{\underline{x^2}}$: $2 = A+B \longrightarrow B = 2-A$

$\underline{\underline{x^1}}$: $-7 = -5B+C \longrightarrow -7 = -5(2-A)+C$

$\underline{1}$: $-14 = A-5C$

$-7 = -10 + 5A + C$

$3 = 5A + C$

$\Downarrow$

$A = -14 + 5C$

$3 = 5(-14+5C) + C$

$3 = -70 + 25C + C$

$A = -14 + 5\left(\frac{73}{26}\right)$

$73 = 26C$

$= \frac{-364 + 365}{26} = \frac{1}{26}$

$\boxed{\frac{73}{26} = C}$

$B = 2-A$

$= 2 - \frac{1}{26}$

$= \frac{52-1}{26} = \frac{51}{26}$

$\boxed{A = 1/26}$

$$\begin{array}{r} {}^{2}14 \\ 26 \\ \hline 84 \\ 28 \\ \hline 364 \end{array}$$

So

$$\int \frac{2x^2-7x-14}{(x-5)(x^2+1)}\,dx = \int \frac{1/26}{x-5} + \frac{51/26\, x + 73/26}{x^2+1}\,dx$$

$$= \frac{1}{26}\cdot \ln|x-5| + \frac{51}{52}\ln|x^2+1| + \frac{73}{26}\arctan(x) + C$$

(2a) $\displaystyle \lim_{x\to 0^+} (1+3x^2)^{x^2} = 1$ $\quad (1+0)^0 = 1$

(2b) $\displaystyle \lim_{n\to\infty} \frac{\ln(n^2+1)}{\sqrt{n}+1} = \lim \frac{\ln(n^2+1)}{n^{1/2}+1} \overset{LH}{=} \lim \frac{\frac{1}{n^2+1}(2n)}{\frac{1}{2}n^{-1/2}+0}$

$$= \lim_{n\to\infty} \frac{\frac{2n}{n^2+1}}{\frac{1}{2n^{1/2}}} = \lim_{n\to\infty} \frac{2n^{3/2}}{n^2+1} = 0$$

(3a) $\displaystyle \sum_{n=1}^{\infty} \frac{n+1}{n+2} - \frac{n}{n+1}$

$$S_1 = \sum_{n=1}^{1}\left(\frac{n+1}{n+2} - \frac{n}{n+1}\right) = \frac{2}{3} - \frac{1}{2} \qquad = \frac{2}{3} - \frac{1}{2}$$

$$S_2 = \sum_{n=1}^{2}\left(\frac{n+1}{n+2} - \frac{n}{n+1}\right) = \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) = \frac{3}{4} - \frac{1}{2}$$

$$S_3 = \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + \left(\frac{4}{5} - \frac{3}{4}\right) = \frac{4}{5} - \frac{1}{2}$$

So $\displaystyle S_n = \frac{n+1}{n+2} - \frac{1}{2}.$ $\quad \displaystyle \lim_{n\to\infty}\left(\frac{n+1}{n+2} - \frac{1}{2}\right) = 1 - \frac{1}{2} = 1/2$

So using telescoping we see the series converges to $1/2$

(3b) $\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n = \left(\frac{5}{6}\right)^1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots$

$= \frac{5}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \cdots\right]$

$= \frac{5}{6} \cdot \frac{1}{1 - 5/6}$

Geometric Series

$|r| = \frac{5}{6} < 1$ So Series

Converges. to

$\frac{5}{6} \frac{1}{1-5/6}$

(3c) $\boxed{\sum_{n=1}^{\infty} \frac{n+1}{n+2}}$

$\lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$, by the divergence test, the series diverges.

(3d) $\boxed{\sum_{n=1}^{\infty} ne^{-n^2}}$

I will use the Integral test (RATIO Test also works).

$\int_1^{\infty} x e^{-x^2}\,dx = \lim_{b\to\infty} \int_1^b x e^{-x^2}\,dx$ $\boxed{\begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array}}$

$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b = \lim_{b\to\infty}\left(\underbrace{-\frac{1}{2e^{b^2}}}_{\to 0} - -\frac{1}{2e^{1^2}}\right) = \frac{1}{2e}$

Since the integral converges, the series converges

$$(3e)\ \boxed{\sum_{n=1}^{\infty} \frac{n^2+1}{2n^3+2}}$$

LCT to $\sum \frac{1}{n}$, by P-series, since $p=1 \le 1$ we have $\sum \frac{1}{n}$ diverges.

$$L = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n^2+1}{2n^3+2}} = \lim \frac{2n^3+2}{n^3+1} = 2$$

$0 < L = 2 < \infty \Rightarrow$ Series diverges since known series diverges.

$$(3f)\ \boxed{\sum \frac{1}{n}}$$

P-Series $p = 1 \le 1 \Rightarrow$ Series diverges

$$(3g)\ \boxed{\sum (-1)^n \frac{1}{n}}$$

AST $\lim \frac{1}{n} = 0$

So by AST Series converges

$$(3h)\ \boxed{\sum \frac{2^n}{n!}}$$

RATIO. $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} 2 \frac{1}{n+1} = 0$ $r = 0 < 1 \Rightarrow$ Series converges

$$(3i)\ \boxed{\sum_{n=1}^{\infty} \left(1 - \frac{3}{n}\right)^{n^2}}$$

Root $r = \lim (a_n)^{1/n} = \lim \left(\left(1-\frac{3}{n}\right)^{n^2}\right)^{1/n}$

$= \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n = e^{-3} = \frac{1}{e}$ $r = \frac{1}{e} < 1$ so series converges