PRACTICE TEST 3

1. $\int \frac{1}{x\ln(x)}\,dx$ $\quad u = \ln(x)$, $\quad du = \frac{1}{x}\,dx$

$$= \int \underbrace{\frac{1}{\ln(x)}}_{u} \cdot \underbrace{\boxed{\frac{1}{x}\,dx}}_{du} = \int \frac{1}{u}\cdot du = \ln|u| + C = \ln|\ln(x)| + C$$

2. $\int \frac{1}{x\,(\ln(x))^2}\,dx$ $\quad u = \ln(x)$, $\quad du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{\ln(x)} + C$$

3. $\int \tan(x)\,dx = \ln|\sec(x)| + C$

4. $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

one of the 16

5. $\int \frac{1}{\sqrt{x^2-4}}\,dx$

$x^2 - 4$

$u - a^2$

$u = x \qquad a = 2$

$u = 2\sec(\theta)$

only used to set up problem
No longer needed

$x = 2\sec(\theta)$

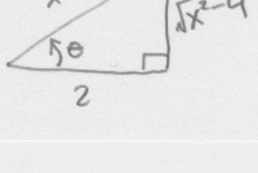
$dx = 2\sec(\theta)\tan(\theta)$

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{\sqrt{(2\sec(\theta))^2-4}}\,2\sec\theta\tan\theta\,d\theta = 2\int \frac{\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}\,d\theta$$

$$= 2\int \frac{}{\sqrt{4}\cdot\sqrt{\sec^2\theta - 1}}\,d\theta = \int \frac{\sec\theta\ \cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

TRIANGLE FROM ORIGINAL SUB

$\frac{x}{2} = \sec\theta = \frac{\text{Hyp}}{\text{Adj}}$



$\sec\theta = \frac{x}{2}$

$\tan\theta = \frac{\sqrt{x^2-4}}{2} \quad \frac{\text{opp}}{\text{adj}}$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

6. $\int \frac{2x+2}{x^2+2x}\,dx$

use partial fractions

$$x\cdot(x+2)\left[\frac{2x+2}{x(x+2)}\right] = \left[\frac{A}{x} + \frac{B}{x+2}\right]x(x+2)$$

$$2x+2 = A(x+2) + Bx$$

$$2x+2 = Ax+2A + Bx$$

$$2x+2 = (A+B)x + 2A$$

$\underline{\underline{x}}$: $2 = A+B$

1: $2 = 2A$

so $A = 1$

AND $B = 1$

$$\int \frac{2x+2}{x^2+2x}\,dx = \int \frac{1}{x} + \frac{1}{x+2}\,dx = \boxed{\ln|x| + \ln|x+2| + C}$$

7. $\int \frac{2x^2+x+1}{x^3+x}\,dx$

use PARTIAL FRACTIONS

$$x(x^2+1)\left(\frac{2x^2+x+1}{x(x^2+1)}\right) = \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right)x(x^2+1)$$

$$2x^2+x+1 = A\cdot(x^2+1) + (Bx+C)x$$

$$2x^2+x+1 = Ax^2 + A + Bx^2 + Cx$$

$$2x^2+x+1 = (A+B)x^2 + Cx + A$$

x^2: $2 = A+B$

x: $1 = C$

1: $1 = A$

$\Rightarrow$ $A=1$, $B=1$ AND $C=1$

So $$\int \frac{2x^2+x+1}{x^3+x}\,dx = \int \frac{1}{x} + \frac{x+1}{x^2+1}\,dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \ln|x| + \frac{1}{2}\ln|x^2+1| + \arctan(x) + C$$

I used u-sub here $u = x^2+1$

8. $\int \frac{2x^2+3x+1}{x(x+1)^2}\,dx$

PARTIAL FRACTIONS

$$\left(\frac{2x^2+3x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right) x\cdot(x+1)^2$$

$x(x+1)^2$

$$2x^2+3x+1 = A(x+1)^2 + Bx(x+1) + C\cdot x$$

$$= A(x^2+2x+1) + Bx^2 + Bx + Cx$$

$$= (A+B)x + (2A+B+C)x + A$$

x^2: $2 = A+B$

x: $3 = 2A+B+C$

1: $1 = A$

$\boxed{A=1}$

$2 = 1+B$

$\boxed{1=B}$

$3 = 2(1)+(1)+C$

$\boxed{0=C}$

So

$$\int \frac{2x^2+3x+1}{x(x+1)^2}\,dx = \int\left(\frac{1}{x} + \frac{1}{x+1} + \frac{0}{(x+1)^2}\right)dx = \ln|x| + \ln|x+1| + C$$

9, 10 we did in class but I'll do #11

11. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx = \lim \int x^{-1/2}\,dx = \lim \frac{x^{1/2}}{1/2}\Big|_1^b$

$$= \lim_{b\to\infty}\left[2(b)^{1/2} - 2(1)^{1/2}\right] = \infty$$ the integral diverges.

(12.) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$] You may use the formula $\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$

or take the ln of both sides

(13.) did in class

(14.) $\lim_{x\to 0^+} (1+x)^{1/x}$ ← Looks similar to #12

$\ln(L) = \lim_{x\to 0^+} \ln\left((1+x)^{1/x}\right)$

$= \lim_{x\to 0^+} \frac{1}{x} \ln(1+x)$

$= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{1+x} = 0$

So $\ln(L) = 0 \Rightarrow \boxed{L = e^0 = 1}$

(15.) For you but very like #14.

(16.) Did in class

(17.) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ use telescoping since $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$

$S_1 = \sum \ln(n+1) - \ln(n) = \ln(2) - \ln(1) = \ln(2)$

$S_2 = \sum \ln(n+1) - \ln(n) = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) = \ln(3)$

$S_3 = \sum \ln(n+1) - \ln(n) = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \ln(4) - \ln(3)) = \ln(4)$

So

$S_N = \ln(N+1)$ Note $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(N+1) = \infty$

So series diverges

(18.) $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

Geometric Series

$$\sum \left(\frac{2}{3}\right)^n = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$$

$a = \frac{2}{3}$ $\quad r = \frac{2}{3}$

Since $|r| = |\frac{2}{3}| < 1$, the series converges to $\frac{a}{1-r} = \frac{2/3}{1-1/3} = \boxed{2}$.

(19.) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

Geometric Series

$r = \frac{3}{2}$ $\quad |r| = \frac{3}{2} \geq 1$ So series diverges

(20.) $\sum_{n=7}^{\infty} e^{-n}$

Geometric Series

$$= e^{-7} + e^{-8} + e^{-9} + e^{-10} + \cdots$$

$a = e^{-7}$ $\quad r = e^{-1} = \frac{1}{e}$

Since $|r| = \frac{1}{e} < 1$ the series converges to $\frac{a}{1-r} = \frac{e^{-7}}{1-e^{-1}} = \frac{1}{e^7 - e^6}$

(21.) $\sum_{n=7}^{\infty} \frac{1}{n \ln(n)}$

INTEGRAL TEST

see Problem #1

$$\int_7^{\infty} \frac{1}{x \ln(x)} dx = \lim_{b \to \infty} \int_7^b \frac{1}{x \ln(x)} dx = \lim_{b \to \infty} \ln(\ln(x)) \Big|_7^b$$

$$= \lim_{b \to \infty} \ln(\ln(b)) - \ln(\ln(7)) = \infty$$

Since the integral diverges the series diverges.

(22) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$

Divergence Test

$\lim a_n = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \frac{1}{e} \neq 0$

So series diverges

(23) $\sum \frac{n^2+1}{3n^2+1}$

Divergence Test

$\lim \frac{n^2+1}{3n^2+1} = \frac{1}{3} \neq 0$

So series diverges

(24) $\sum_{n=1}^{\infty} \frac{1}{n}$

P-series

$P = 1 \leq 1$ So series diverges

(25) $\sum \frac{1}{n^2}$ AND (26) $\sum \frac{1}{\sqrt{n}}$ are both P-series AND for you

(27) $\sum \frac{n^2+1}{n^3+1}$

LCT

KNOWN $\sum \frac{1}{n}$ diverges by p-series $(p=1 \leq 1)$

$L = \lim \frac{\frac{1}{n}}{\frac{n^2+1}{n^3+1}} = \lim_{n\to\infty} \frac{n^3+1}{n^3+n} = \underset{0<L<\infty}{1}$. So $\sum \frac{n^2+1}{n^3+1}$ diverges Since $\sum \frac{1}{n}$ diverges

(28) $\sum \frac{n^2+1}{n^4+1}$

LCT to $\sum \frac{1}{n^2}$

which converges since $p = 2 > 1$ by P-series

$L = \lim \frac{\frac{n^2+1}{n^4+1}}{\frac{1}{n^2}} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^4+n^2}{n^4+1} = 1$

Since $0 < 1 < \infty$ both Series converge.

NOTICE NUMERATOR & DENOMINATOR CAN SWITCH

(29) $\sum \frac{\sqrt{n+5}}{n^2+1}$ LCT to $\sum \frac{1}{n^{3/2}}$ which converges by P-Series Test since $p = \frac{3}{2} > 1$.

$$L = \lim_{n\to\infty} \frac{\frac{\sqrt{n+5}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim \frac{\sqrt{n+5}\cdot n^{3/2}}{n^2+1} = \lim \frac{\sqrt{n^4+5n^3}}{n^2+1} = 1$$

$\frac{\sqrt{n^4}}{n^2} = 1$

Since $0 < L = 1 < \infty$, we have both series converge.

(30.) $\sum \frac{2^n+n^2}{3^n+n^3}$ LCT to $\sum \frac{2^n}{3^n}$ which converges by Geometric Series (#18)

$$L = \lim \frac{\frac{2^n+n^2}{3^n+n^3}\cdot\frac{3^n}{2^n}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{6^n + n^2\cdot 3^n}{6^n + n^3\cdot 2^n} = 1$$

Since $0 < L = 1 < \infty$ both series converge.

(31) $\sum (-1)^n \frac{1}{n}$ AST $\lim \frac{1}{n} = 0$

So series converges

(32) $\sum (-1) \frac{1}{n^2}$ AST $\lim \frac{1}{n^2} = 0$

So series converges

Compare to #24, #25 AND #26

(33) $\sum (-1) \frac{1}{\sqrt{n}}$ AST $\lim \frac{1}{\sqrt{n}} = 0$

So series converges

(34) $\sum_{n=1}^{\infty} \frac{n}{n!}$

RATIO TEST

$$r = \lim \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{(n+1)!}}{\frac{n}{n!}} = \lim \frac{n+1}{n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{1}{n} = 0$$

$|r| = 0 < 1$ So Series Converges

(35.) $\sum \frac{e^n}{n!}$

RATIO TEST

$$r = \lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{e^n} \cdot e^1}{\cancel{e^n}} \cdot \frac{\cancel{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}}{(n+1)\cancel{(n)(n-1)\cdots 3\cdot 2\cdot 1}}$$

$$= \lim_{n\to\infty} \frac{e}{n} = 0$$

$r = 0 < 1$ So series converges

(36) $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

RATIO TEST

$$r = \lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \lim \frac{(n+1)^2}{n^2} \cdot \frac{e^n}{e^{n+1}}$$

$$= \lim \left(\frac{n+1}{n}\right)^2 \cdot \frac{e^n}{\cancel{e^n} \cdot e^1} = (1) \cdot \frac{1}{e} = \frac{1}{e}$$

$|r| = \frac{1}{e} < 1 \Rightarrow$ Series converges

(37.) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

Diversence test

$$\lim_{n\to\infty} \frac{e^n}{n^2} = \infty \neq 0$$

So series diverges

(38.) $\displaystyle\sum \frac{n^n}{n!}$

Divergence Test

$$\lim \frac{n^n}{n!} = \infty \neq 0$$

So Series diverges

(39.) $\displaystyle\sum \frac{n!}{n^n}$

we did in class, Recall we got

$r = \frac{1}{e} < 1$ series converges

(40.) $\displaystyle\sum \left(\frac{n^2+1}{3n^2+1}\right)^n$

ROOT TEST

$$r = \lim \left(\left(\frac{n^2+1}{3n^2+1}\right)^n\right)^{1/n}$$

$$= \lim \frac{n^2+1}{3n^2+1} = \frac{1}{3} < 1 \quad \text{Series converges}$$

41 & 42 Similar to 40

41 $\Rightarrow$ diverge with $r = \frac{3}{2}$ AND 42 converge with $r = \frac{2}{3}$

(43.) $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

ROOT TEST

$$r = \lim (\ a_n\ )^{1/n} = \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$$

$|r| = e^{-1} < 1$ So series converges.

Note similarity with #43 AND #22.

ALSO NOTE THE DIFFERENCES.