Test 2 PART 2

(1.) $$\boxed{\int \frac{6x^3+x^2+x}{x^4-1}\,dx}$$

Note $x^4-1=(x^2+1)(x^2-1)$
$=(x^2+1)(x-1)(x+1)$

So
$$\frac{6x^3+x^2+x}{x^4-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1} \Big) \cdot (x^2+1)(x-1)(x+1)$$

$$\boxed{6x^3+x^2+x = (Ax+B)(x-1)(x+1) + C(x+1)(x^2+1) + D(x-1)(x^2+1)}$$

$$=(Ax+B)(x^2-1) + C(x^3+x^2+x+1) + D(x^3-x^2+x-1)$$

$$= Ax^3 - Ax + Bx^2 - B + Cx^3 + Cx^2 + Cx + C + Dx^3 - Dx^2 + Dx - D$$

$$= x^3(A+C+D) + x^2(B+C+-D) + x(-A+C+D) + (-B+C-D)$$

$\underline{x^3}$: $6 = A+C+D$

$\underline{x^2}$: $1 = B+C-D$

$\underline{x^1}$: $1 = -A+C+D$

$\underline{1}$: $0 = -B+C-D$

plug in $x=1$

$$6+1+1 = (A+B)(0) + C(1+1)(1^2+1) + D(0)(1+1)$$

$$8 = 4C \Rightarrow \boxed{C=2}$$

Plug in $x=-1$

$$-6+1-1 = (-A+B)(0) + C(-1+1)(1+1) + D(-1-1)(1+1)$$

$$-6 = D(-4) \Rightarrow \boxed{D=\tfrac{3}{2}}$$

$6 = A+C+D$
$6 = A+2+\frac{3}{2}$
$2\tfrac{1}{2} = A$
$\boxed{\frac{5}{2}=A}$

$1 = B+C+-D$
$1 = B+2-\frac{3}{2}$
$\boxed{\tfrac{1}{2}=B}$

1. (CONTINUED) So

$$\int \frac{6x^3+x^2+x}{x^4-1}\,dx = \int \frac{\frac{5}{2}x+\frac{1}{2}}{x^2+1}\,dx + \int \frac{2}{x-1}\,dx + \int \frac{3/2}{x+1}\,dx$$

$$= \frac{5}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx + 2\ln|x-1| + \frac{3}{2}\ln|x+1| + C$$

$$= \frac{5}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}(x) + 2\ln|x-1| + \frac{3}{2}\ln|x+1| + C$$

2. $\boxed{\sum_{k=1}^{\infty} \frac{k^2}{3^k}}$ Use Ratio Test

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^2}{3^{k+1}} \cdot \frac{3^k}{k^2}}{\frac{k^2}{3^k}} = \lim \frac{(k+1)^2}{k^2} \cdot \frac{3^k}{3^{k+1}} = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

$\rho = \frac{1}{3} < 1 \Rightarrow$ series converges

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3. $\sum_{k=1}^{\infty} \left[\frac{3k^2+3k+1}{4k^2+1}\right]^k$ ← HINT use Root Test

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\left[\frac{3k^2+3k+1}{4k^2+1}\right]^k} = \lim_{k\to\infty} \frac{3k^2+3k+1}{4k^2+1}$$

$$= \frac{3}{4}$$

$\rho = 3/4 < 1 \Rightarrow$ Series converges

4. $f(x) = \ln(x+3)$ $\quad n=4 \quad x_0 = 0$. Estimate $\ln(3.01)$

$f(x) = \ln(x+3)$ $\qquad f(0) = \ln(3)$

$f'(x) = \frac{1}{x+3} = (x+3)^{-1}$ $\qquad f'(0) = \frac{1}{3}$

$f''(x) = -(x+3)^{-2}$ $\qquad f''(0) = -\frac{1}{9}$

$f'''(x) = 2(x+3)^{-3}$ $\qquad f'''(0) = \frac{2}{3^3}$

$f^{(4)}(x) = -6(x+3)^{-4}$ $\qquad f^{(4)}(0) = \frac{-6}{3^4}$

$$P_4(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4$$

$f'(0) = 1/3 \qquad x_0 = 0$

$$\boxed{P_4(x) = \ln(3) + \frac{1}{3}x - \frac{1}{9(2!)}x^2 + \frac{2}{3^3(3!)}x^3 - \frac{6}{3^4(4!)}x^4}$$

$$\boxed{P_4(0.01) = \ln(3) + \frac{1}{3}(0.01) - \frac{1}{9(2!)}(0.01)^2 + \frac{2}{3^3(3!)}(0.01)^3 - \frac{6}{3^4 \cdot 4!}(0.01)^4}$$