Completing the square

1. Preliminaries

We learned about completing the square in some previous algebra class. We will start by reminding you with a few simple problems. Completing the square (CTS) is a technique that can be used for a few purposes. Initially, we will look at using it to solve a quadratic equation. The following quadratic equation can be solved with the quadratic formula. We will use CTS instead.

$$x^2 + 8x + 11 = 0 \tag{1}$$

First we will add and subtract the same quantity - 16 in this problem.

$$x^2 + 8x + 16 - 16 + 11 = 0$$

The note that $x^2 + 8x + 16$ can be factored as a square.

$$(x^2 + 8x + 16) - 16 + 11 = 0$$
$$(x + 4)^2 - 16 + 11 = 0$$
$$(x + 4)^2 - 7 = 0$$

Then solve for the square

$$(x + 4)^2 = 7$$
$$(x + 4) = \pm\sqrt{7}$$
$$x = -4 \pm \sqrt{7}$$

And we have solved the quadratic. The solutions to equation 1 are $x = -4+\sqrt{7}$ and $x = -4 - \sqrt{7}$.

How did I know to add ad subtract 16? Simply take divide the x coefficient by 2 and square. So in this problem we took 8. Divided by 2 to get 4. And the square 4 to get 16. Add and subtract 16.

Problem 1. Use CTS to $x^2 + 18x + 10 = 0$.

How do we handle it if the coefficient in front of the square term is not one? There area a variety of ways. See the following technique.

$$4x^2 + 8x + 11 = 0 \tag{2}$$

$4x^2 + 24x + 11 = 0$

$4(x^2 + 6x) + 11 = 0$ factor out the four

$4(x^2 + 6x + 9 - 9) + 11 = 0$ the divide 6 by 2 and square to get 9. Add and subtract a 9

$4(x^2 + 6x + 9) - 4(9) + 11 = 0$ distribute the 4

$4(x^2 + 6x + 9) - 4(9) + 11 = 0$ distribute the 4

$4(x^+3)^2 - 4(9) + 11 = 0$ factor our square

$4(x^+3)^2 - 36 + 11 = 0$ simplify

$4(x^+3)^2 - 25 = 0$

$4(x^+3)^2 = 25$

$(x^+3)^2 = 25/4$

$(x^+3) = \pm 5/2$

$x = -3 \pm 5/2$

So the solutions to equation 2 are $x = -3 + 5/2 = -1/2$ and $x = -3 - 5/2 = -11/2$

Problem 2. Use CTS to $3x^2 + 18x + 10 = 0$.

2. Using Completing the square in Calculus

This is a calculus course and so we will use CTS in a calculus problem.

$$\int \frac{1}{(x^2+4x+5)^{3/2}}\,dx$$

For this problem we CTS the denominator and then use a trigonometric substitution.

$$\begin{aligned}
x^2+4x+5 &= x^2+4x+4-4+5\\
&= (x^2+4x+4)-4+5\\
&= (x+2)^2-4+5\\
&= (x+2)^2+1
\end{aligned}$$

So the integral becomes

$$\int \frac{1}{(x^2+4x+5)^{3/2}}\,dx = \int \frac{1}{((x+2)^2+1)^{3/2}}\,dx$$

Here we use a trig substitution of the form $u = a\tan(\theta)$ where $u = x+2$ and $a = 1$.

So we have

$$x+2 = \tan(\theta) dx = \sec^2(\theta)d\theta$$

So

$$\begin{aligned}
\int \frac{1}{(x^2+4x+5)^{3/2}}\,dx &= \int \frac{1}{((x+2)^2+1)^{3/2}}\,dx\\
&= \int \frac{1}{((\tan(\theta))^2+1)^{3/2}}\sec^2(\theta)d\theta\\
&= \int \frac{\sec^2(\theta)}{((\sec(\theta))^2)^{3/2}}\,d\theta\\
&= \int \frac{\sec^2(\theta)}{\sec^3(\theta)}\,d\theta\\
&= \int \frac{1}{\sec(\theta)}\,d\theta\\
&= \int \cos(\theta)\,d\theta\\
&= \sin(\theta)+C
\end{aligned}$$

Set up the triangle with $\tan(\theta) = \frac{x+2}{1}$. And get that $\sin(\theta) = \frac{x+2}{\sqrt{(x+2)^2+1}}$.

So

$$\int \frac{1}{(x^2+4x+5)^{3/2}}\,dx = \sin(\theta)+C = \frac{x+2}{\sqrt{(x+2)^2+1}}+c$$

Problem 3. Compute the following integral (CTS will be useful).

$$\int \frac{1}{x^2 + 6x + 18} \, dx$$

Problem 4. Compute the following integral (CTS will be useful).

$$\int \frac{1}{(x^2 - 8x - 9)^{3/2}} \, dx$$

For this problem I got $u = x - 4$ and $a = 5$ and $u = a\sec(\theta)$