Taylor Series, Euler's Formula and /pi

1. Euler's Formula

One of the most theorems in mathematics is Euler's Formula. It can relate the trigonometric functions the exponential function and the imaginary number, i.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This equation is a beautiful confluence of different branches of mathematics you have learned and has numerous theoretic and practical uses.

First recall the mathematics of the imaginary number, i. Recall $i^2 = -1$.

So $i^3 = i^2 i = -1i = -i$. And when we multiply complex numbers (like $a + ib$) we use some distribution rule (or FOIL). As in

$$\begin{aligned}(3-2i)(4+5i) &= 3\cdot 4 + 3\cdot(5i) - 2i\cdot 4 - 2i\cdot 5i \\ &= 12 + 15i - 8i - 10i^2 \\ &= 12 + 15i - 8i + 10 \text{ since } i^2 = -1 \\ &= 22 + 7i.\end{aligned}$$

We always like to group the real numbers and the imaginary numbers.

Problem 1. Simplify the following.

(1) i^4
(2) i^5
(3) i^{105}
(4) $(2+i)(4-3i)$
(5) $(1+i)^4$

The last one is interesting. We get $(1+i)^4 = -4$. So is that like saying $\sqrt[4]{-4} = 1 + i$?

Problem 2. We will Prove (verify really) the formula using Taylor series.

- Write down the series e^x and substitute $x = i\theta$.
- We now have the Taylor series for $e^{i\theta}$.
- Simplify your series. That is turn things like this $(i\theta)^3$ into this $-i\theta^3$.
- Group your terms with i and terms without i. Factor out the i and write as follows
$$e^{i\theta} = (1 + \cdots) + i(x + \cdots).$$
- You should recognize the series inside of the parenthesis, substitute. What do you have?

2. How do we find π?

There are many ways to use a series to compute digits of π. We will use a very simple method. Recall that $\arctan(1) = \pi/4$, so $4\arctan(1) = \pi$.

Recall the series for $\frac{1}{1-x}$. We will perform a few steps to this series and the have a series equal to π we can compute.

Problem 3.

- Write down the series $\frac{1}{1-x}$ and substitute $-x^2$ for x to get the series for $\frac{1}{1+x^2}$.
- Integrate the equation with the series for $\frac{1}{1+x^2}$. This will give you $\arctan(x) = \cdots$.
- Multiply both sides by 4 to get $4\arctan(x) = \cdots$.

- Substitute 1 in for x to get $4\arctan(1) = \cdots$. Add up your first five terms. What do you get? Is it close to π?

This series converges very slowly to π. That is, we need many terms to get a few decimal places of accuracy. The incredible[1] mathematician Srinavasa Ramanujan found a better (converges quicker) formula in

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{26390k + 1103}{396^{4k}}.$$

[1]Here I say incredible and mean it. Ramanujan's mathematics was amazing and certainly qualifies as literally incredible. You can read about his life in the book **The Man Who Knew Infinity** by Robert Kanigel or even see a movie about him called **The Man Who Knew Infinity** from 2015. I know how cool is it that there are math movies!