Math 6250: Final Exam Review

1 Injective, Surjective

1. Define $f : (0, \infty) \to (0, 1)$ by $f(x) = \frac{x}{1+x}$.
   (a) Graph this function.
   (b) Prove f is injective.
   (c) Prove f is surjective.

2 Cardinality

2. What is the definition A and B have the same cardinality.
3. What is the definition A is countable.
4. List five sets which are countably infinite. List three sets that are uncountable.
5. Show the following sets have the same cardinality
   (a) $\mathbb{N}$ and $\mathbb{Z}$
   (b) $\mathbb{N}$ and $\mathbb{Q}$
   (c) $(0, \infty) \sim (0, 1)$.
   (d) $(2, 4] \sim [3, 7)$.

3 Some Complex Questions

6. Find all $z \in \mathbb{C}$ so that
   - $z^3 = 1$
   - $z^3 = i$
   - $z^2 = i$
   - $z^4 = -1$
   - $z^4 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$
   - $z^3 = -1$

7. Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$. Graph z, z^2, z^3. Describe what happens graphically when we square z. Look at the argument.

8. Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $w = i$. Graph z, w and zw. Describe what happens graphically when we multiply z and w. Look at the argument.

9. Use Euler's equation to prove a familiar trigonometric identity for
$$\cos(3\alpha).$$

10. Use Euler's equation to prove a familiar trigonometric identity for
$$\sin(\alpha + \beta).$$

4 Limits of Sequences

11. Compute the following limits and use the $\varepsilon - N$ definition to prove it.
    (a) $\lim_{n\to\infty} \frac{1}{3n^2 + 5n + 1}$
    (b) $\lim_{n\to\infty} \frac{n}{3n + 1}$
    (c) $\lim_{n\to\infty} \frac{3n + \sin(n)}{3 - 4n^2}$

12. Use the $\varepsilon - N$ definition to prove If (a_n) is a convergent sequence then (a_n) is bounded.

13. Use the $\varepsilon - N$ definition to prove
    (a) If $\lim a_n = a$ and $k \in \mathbb{R}$ then $\lim ka_n = ka$
    (b) If $\lim a_n = a$ and $\lim b_n = b$ then $\lim a_n + b_n = a + b$
    (c) If $\lim a_n = a$ and $\lim b_n = b$ then $\lim a_n b_n = ab$

14. For the following questions use the Monotone Convergence Theorem and the sequence
$$a_1 = 1 \text{ and } a_{n+1} = 2 - \frac{1}{a_n + 1}.$$
    - Write the first four terms of the series.

- Show (a_n) is monotone.
- Show (a_n) is bounded.
- Prove (a_n) is convergent.
- What is the limit of (a_n).

5 Limits of Functions

15. Compute the following limits and use the $\varepsilon - \delta$ definition to prove it.

    (a) $\lim_{x\to 3} x^2 + 2x$

    (b) $\lim_{x\to c} x^2 + 2x$

    (c) $\lim_{x\to 9} \sqrt{x} = 3$

16. Use the $\varepsilon - \delta$ definition to prove

    (a) If $\lim_{x\to c} f(x) = F$ and $k \in \mathbb{R}$ then ${}_{x\to c} kf(x) = kF$

    (b) If $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ then $\lim_{x\to c} f(x) + g(x) = F + G$

17. Use the fact that
$$\lim_{a\to 0} \frac{\sin(a)}{a} = 1$$
    to solve the following (do not use $\varepsilon - \delta$):

    (a) $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

    (b) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

    (c) $\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$

    (d) $\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$. Hint use Problem 10.

6 Continuity

18. Show the following functions are continuus (or not).

    (a) $f(x) = x^3$ at $x = c$

    (b) $f(x) = \frac{1}{x}$ at $x = c$

    (c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$

(f) $f(x) = \frac{x^4+x^2}{|x|}$ at $x = 0$

(g) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(h) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(i) $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(j) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(k) $f(x) = \begin{cases} x^3\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

7 Derivatives

19. Prove if f is differentiable at $x = c$ then f is continuous at $x = c$.

20. From the definition compute the derivatives for the following functions.

(a) $f(x) = x^3$ at $x = c$

(b) $f(x) = \frac{1}{x}$ at $x = c$

(c) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(d) $f(x) = \frac{x^2+x}{|x|}$ at $x = 0$

(e) $f(x) = \frac{x^3+x^2}{|x|}$ at $x = 0$

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(k) $f(x) = \begin{cases} x^3 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$

21. Recall that

$$\lim_{\alpha \to 0} \frac{\sin(\alpha)}{\alpha} = 1.$$

Use this fact to prove the following

(a) $\lim_{\alpha \to 0} \frac{1-\cos(\alpha)}{\alpha} = 0.$

(b) $\lim_{\alpha \to 0} \frac{\sin^2(\alpha)}{\alpha^2} = 1.$

(c) $\lim_{\alpha \to 0} \frac{1-\cos^2(\alpha)}{\alpha^2} = 0.$

(d) $\lim_{\alpha \to 0} \frac{[1-\cos(\alpha)]^2}{\alpha^2} = 0.$

(e) $\lim_{\alpha \to 0} \frac{\tan(\alpha)}{\alpha} = 1.$

22. Note for $f(x) = \sin(x)$ we have shown

$$\lim_{\alpha \to 0} \frac{1 - \cos(\alpha)}{\alpha} = 0.$$

And recall the trigonometric identity from Problem 17d Compute the derivative of $f(x)$ using these two facts.

8 Trigonometry and Dimensions

23. Compute the Taylor series for the following functions at $c = 0$.

(a) $f(x) = (x-3)^4$

(b) $f(x) = \sin(2x)$

(c) $f(x) = e^{5x}$