Polynomial Approximations

1 Preliminaries

Before we start this exercise let's recall an old problem from Calculus I. Find the tangent line to $f(x) = x^2 - 6x + 10$ at the point $x_0 = 0$.
Solution: First note $f'(x) = 2x - 6$. Now we compute the point and the slope.
Point: We have $x_0 = 0$ so $y_0 = f(0) = 0^2 - 6 \cdot 0 + 10 = 10$. So our point is $(x_0.y_0) = (0, 10)$.
Slope: We have $x_0 = 0$ so $m = f'(0) = 2(0) - 6 = -6$. So our slope is $m = -6$.
Now we put together for our line.

$$y - y_0 = m(x - x_0)$$

$$y - 10 = -6(x - 0)$$

$$y = -6x + 10$$

So we have $y = -6x + 10$. Let's call it $P(x) = -6x + 10$.

Notice our tangent line has some nice properties.

1. $f(0) = P(0)$
2. $f'(0) = P'(0)$
3. and because of 1 and 2 for x-values near $x_0 = 0$ we have that $f(x) \approx P(x)$. See table below.

x	$f(x)$	$P(x)$	Comment
-0.1	10.61	10.6	very close
0	10	10	exactly equal
0.1	9.41	9.4	very close
1	5	4	pretty close
2	2	-2	not really close
3	1	-8	far off

So our tangent line (which we called $P(x)$) is an approximation of the original function $f(x)$ for x-values near 0.

2 Polynomials

For this exercise we will approximate a function with a polynomial. Polynomials are simpler than many other functions (for example $\sin(x)$ and e^x). The polynomials we will consider are:

$$P_1(x) = A_0 + A_1x^1$$

$$P_2(x) = A_0 + A_1x^1 + A_2x^2$$

$$P_5(x) = A_0 + A_1x^1 + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5$$

Or any other order. We say the highest power of the polynomial is its order. So the order of $P_1(x)$ is 1 and the order of $P_5(x)$ is 5.

2.1 First Order

$$P_1(x) = A_0 + A_1 x^1$$

We will use the tangent line example as our guide. We will want to satisfy

1. $f(0) = P_1(0)$, and
2. $f'(0) = P_1'(0)$.

So given a function $f(x)$ we want to find $P_1(x)$, that is, we want to find A_0 and A_1 satisfying 1 and 2.

First let's satisfy: $f(0) = P_1(0)$. We know that $P_1(0) = A_0 + A_1(0) = A_0$. So using this and $f(0) = P_1(0)$ we get $A_0 = f(0)$. Remember we are trying to find A_0 and A_1 so far so good.

Now let's satisfy: $f'(0) = P_1'(0)$. We know that $P_1'(x) = A_1$. So $P_1'(0) = A_1$. Using this and $f'(0) = P_1'(0)$ we get $A_1 = f'(0)$.

So

$$A_0 = f(0)$$
$$A_1 = f'(0)$$

Let's use this on an example. Say $f(x) = x^2 - 6x + 10$ Our first derivative is $f'(x) = 2x - 6$. So

$$A_0 = f(0) = 0^2 - 6(0) + 10 = 10$$

and

$$A_1 = f'(0) = 2(0) - 6 = -6$$

So our first order polynomial is $P_1(x) = 10 - 6x^1$ Looks identical to what we did in Section 1.

2.2 Third Order

$$P_3(x) = A_0 + A_1 x^1 + A_2 x^2 + A_3 x^3$$

Now we will approximate a given function $f(x)$ using a third order polynomial. We should be able to find an even better approximate. For this we will want to satisfy

1. $f(0) = P_3(0)$, and
2. $f'(0) = P_3'(0)$.
3. $f''(0) = P_3''(0)$.
4. $f'''(0) = P_3'''(0)$.

So given a function $f(x)$ we want to find $P_3(x)$, that is, we want to find A_0, A_1, A_1 and A_3 satisfying 1, 2, 3 and 4.

First let's satisfy: $f(0) = P_3(0)$ We know that $P_3(0) = A_0 + A_1(0)^1 + A_2(0)^2 + A_3(0)^3 = A_0$. So using this and $f(0) = P_3(0)$ we get $A_0 = f(0)$.

Now let's satisfy: $f'(0) = P_3'(0)$. We know that $P_3'(0) = A_1 + 2A_2 x + 3A_3 x^2$. So $P_3'(0) = A_1 + 2A_2(0) + 3A_3(0)^2 = A_1$. So using this and $f'(0) = P_3'(0)$ we get $A_1 = f'(0)$.

So

$$A_0 = f(0)$$
$$A_1 = f'(0)$$

Same as before now to find, A_2 and A_3 satisfying the two new conditions. Now let's satisfy: $f''(0) = P_3''(0)$. We know that $P_3''(0) = 2A_2 + 6A_3x$. So $P_3''(0) = 2A_2 + 3A_3(0) = 2A_2$. So using this and $f''(0) = P_3''(0)$ we get $2A_2 = f''(0)$ thus $A_2 = \frac{1}{2}f''(0)$.

Now we have

$$A_0 = f(0)$$
$$A_1 = f'(0)$$
$$A_2 = \frac{1}{2}f''(0)$$

And lastly we need to satisfy $f'''(0) = P_3'''(0)$. We know that $P_3'''(0) = 6A_3$. So $P_3'''(0) = 6A_3$. So using this and $f'''(0) = P_3'''(0)$ we get $6A_3 = f'''(0)$ thus $A_3 = \frac{1}{6}f'''(0)$.

Our summary:

$$A_0 = f(0)$$
$$A_1 = f'(0)$$
$$A_2 = \frac{1}{2}f''(0)$$
$$A_3 = \frac{1}{6}f'''(0)$$

So know we have a polynomial that satisfies 1, 2, 3 and 4. So it should approximate $f(x)$ even better.

Example: I will approximate $f(x) = \sin(x)$ and compute $P_3(x)$. First I will compute the first three derivatives of $f(x)$ and then plug them using

$$A_0 = f(0)$$
$$A_1 = f'(0)$$
$$A_2 = \frac{1}{2}f''(0)$$
$$A_3 = \frac{1}{6}f'''(0)$$

- $A_0 = f(0) = \sin(0) = 0$
- First compute $f'(x) = \cos(x)$. So $A_1 = f'(0) = \cos(0) = 1$
- Now compute $f''(x) = -\sin(x)$. So $A_2 = \frac{1}{2}f''(0) = -\frac{1}{2}\sin(0) = 0$
- Now compute $f'''(x) = -\cos(x)$. So $A_3 = \frac{1}{6}f'''(0) = -\frac{1}{6}\cos(0) = \frac{-1}{6}$.

So plugging into $P_3(x) = A_0 + A_1x^1 + A_2x^2 + A_3x^3$ we get

$$P_3(x) = 0 + 1x^1 + 0x^2 + \frac{-1}{6}x^3 = x + \frac{-x^3}{6}x^3$$

So $P_3(x)x + \frac{-x^3}{6}x^3 \approx \sin(x)$ near $x = 0$. I graphed a few below

Notice in Figure 2 that the $P_5(x)$ is a better approximation to $\sin(x)$ than it either $P_1(x)$ or $P_3(x)$.

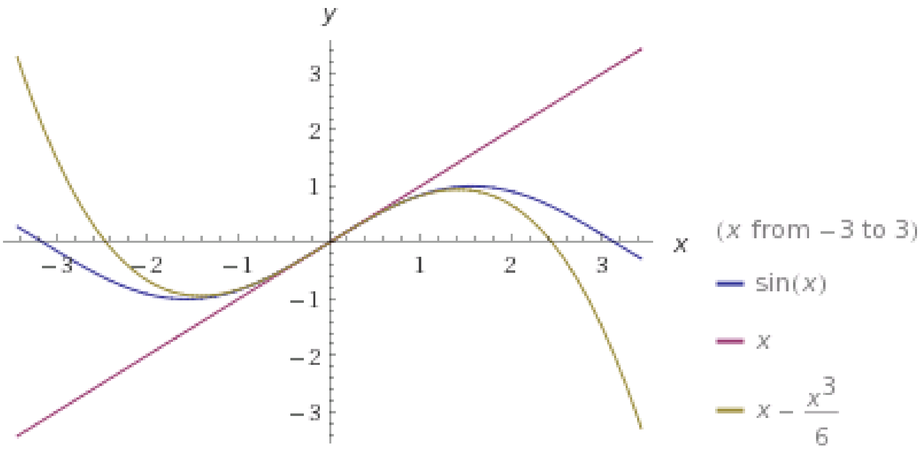

Figure 1: Both $P_1(x)$ and $P_3(x)$ graphed with $\sin(x)$

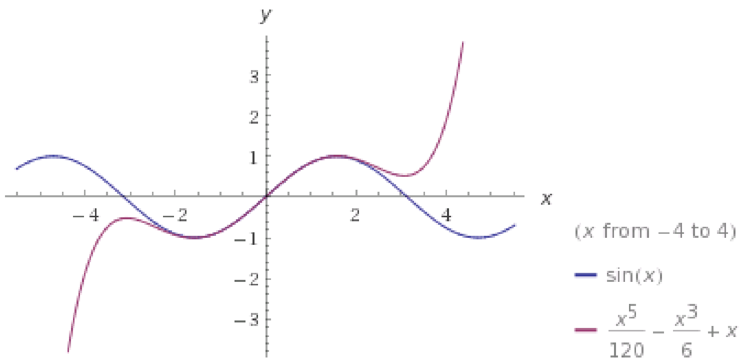

Figure 2: $P_5(x)$ graphed with $\sin(x)$

2.3 Fifth Order and Seventh

I have spoken long enough. Now for you to speak.

1. Find the equations for $A_0, A_1, \ldots, A_5$ for

$$P_5(x) = A_0 + A_1 x^1 + A_2 x^2 + A_3 x^3 + A_4 x^4 + A_5 x^5$$

   like we did for $P_1(x)$ and $P_3(x)$.

2. Use your formulas to find the $P_5(x)$ approximation for

   (a) $f(x) = e^x$

   (b) $f(x) = \cos(x)$

   (c) $f(x) = \ln(1 + x)$

3. Guess the formulas for for $A_0, A_1, \ldots, A_7$ for $P_7(x)$. Hint: $A_0, A_1, \ldots, A_5$ should be identical to what you did for $P_5(x)$, so you need to only guess A_6 and A_7.