Math 3160 - Final Exam Review

Know **Test 1 Review** and **Test 2 Review** and this review.

1 Diagonalize

1. For the following matrices find the characteristic equation, the eigenvalues and their corresponding eigen vectors.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

and

$$E = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix},$$

2. for the above matrices, determine if they are diagonalizable. State why or why not. And if it is diagonalizable, diagonalize it. That is, find P and D.

3. Diagonalize the matrix below.

$$\begin{bmatrix} 4 & 0 & -1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

2 Hill Cipher

4. Solve the following for x. Answers should be reduced to

$$\{0, 1, 2, \ldots, n-1\}.$$

   (a) $3x \equiv 1 \mod 7$

   (b) $3x \equiv 1 \mod 9$

   (c) $3x \equiv 4 \mod 7$

   (d) $3x \equiv 6 \mod 9$

   (e) $x^2 \equiv 1 \mod 7$

(f) $x^2 \equiv 1 \mod 9$

(g) $11x \equiv 1 \mod 26$

(h) $14x \equiv 1 \mod 26$

5. For the E below find D. $E = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

6. For the E defined in 5 encipher the plaintext message "FISH".

7. For the E defined in 5 decipher the ciphertext message "ICQV".

8. For the E below find D, encipher the plaintext message "VIN" and decipher the ciphertext message "JLEU".

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

9. For the E below find D. The Hill cipher for this type of matrix is called a **symmetric cipher**. What do you think a symmetric cipher is? Examples of symmetric ciphers include the enigma machine and DES. What does this mean for enciphering and deciphering?

$$E = \begin{bmatrix} 12 & 1 \\ 1 & 13 \end{bmatrix}$$

3 Orthogonalization

10. Let $\mathbf{v_1} = (1,2,0)$, $\mathbf{v_2} = (1,0,0)$, $\mathbf{v_3} = (-1,0,1)$. Orthogonalize.

11. Let $\mathbf{p_1} = 1$, $\mathbf{p_2} = x - x + 1$, $\mathbf{p_3} = x$. Orthogonalize.

12. For the following consider the inner product space on $C[0,1]$ with inner product defined as

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

(a) Let $\mathbf{p_1} = 1$, $\mathbf{p_2} = x - x + 1$, $\mathbf{p_3} = x$. Orthogonalize.

(b) Let $\mathbf{f_2} = e^x$, $\mathbf{f_2} = e^{2x}$, $\mathbf{p_3} = x$. Orthogonalize.

13. Assume $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^n$ satisfy the following properties

- $\mathbf{u}_1 \cdot \mathbf{u}_1 = 1$, $\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$ and $\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$
- $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_3 \cdot \mathbf{u}_2 = 0$

Show the change of basis matrix $P_{\text{STANDARD}\to B}$ is given by the matrix with rows $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V with the following properties:

    - $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$ and $\mathbf{v}_3$, and $\mathbf{v}_2$ is perpendicular to $\mathbf{v}_3$
    - the norm of each vector is 1.

    (a) What is $\mathbf{v}_2 \cdot \mathbf{v}_3$?

    (b) What is the norm of $\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3$?

    (c) What is the angle between $\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3$ and $\mathbf{v}_2$?

    (d) What is the norm of $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$ where $a, b, c \in \mathbb{R}$?