Math 3160 - Final Exam

Name:____

No calculators or electronic devisces of any kind and show all work.

1. Find the solution to the given linear system.

$$\begin{cases} x_1 & +2x_2 & +3x_3 & -2x_4 & = 4 \\ x_1 & & & +x_4 & = 0 \\ -x_1 & & -x_3 & +3x_4 & = 6 \end{cases}$$

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2. Finish the following definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if and only if

3. Let $\mathbf{v}_1 = (1, 0, 2, -1)$, $\mathbf{v}_2 = (1, 1, 0, 1)$ and $\mathbf{v}_3 = (3, 2, 2, 1)$. Show the list of vectors is NOT linearally independent by finding a non-trivial linear combination of the vectors equal to zero.

4. Let $P(1, 0, 4)$, $Q(0, 3, 0)$ and $R(0, 3, 4)$ be points in $\mathbb{R}^3$.

   (a) Compute the area of the triangle formed by the points P, Q and R.

   (b) What is the standard equation of the plane containing the triangle from Problem 4a?

   (c) What is the parametric (or vector) equation of the plane containing the triangle from Problem 4a?

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5. Let $V = \{(x, y, z) \in \mathbb{R}^3 | x + 3y - z = 0\}$. Prove V is a subspace of $\mathbb{R}^3$.

6. The linear transformation is given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 \\ -x_1 + 3x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ 2x_2 \end{bmatrix}.$$

(a) Find the matrix, A, to represent the linear trasformation T.

(b) Compute the basis for the Range of T.

(c) Find a basis for the null space of A, NULL(A).

(d) Find the rank and nullity.

(e) What is the dimension of the domain of T and the codomain of T? Compare Rank, Nullity and the dimension of the Domain.

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7. Let a sequence be defined by the following recursive formula

$$a_1 = 1, a_2 = 2 \text{ and } a_{n+2} = a_{n+1} + 6a_n$$

(a) Compute the first five terms of the sequence.

(b) Find a matrix A for the sequence as we did in class.

(c) Diagonalize A. That is find D and P so that $A = PDP^{-1}$.

(d) Compute $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using Problem 7c.

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8. Define the follwing matrix: $E = \begin{bmatrix} 4 & 3 \\ 7 & 3 \end{bmatrix}$

   (a) For the plaintext message "DOGS", find the two letter block representation using the alphabet: $A = 00, B = 01, C = 02, D = 03, \ldots$

   (b) Use the matrix E to encipher the plaintmessage.

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9. Define the follwing matrix: $E = \begin{bmatrix} 4 & 3 \\ 7 & 3 \end{bmatrix}$

(a) Find the deciphering matrix D.

(b) Decipher the ciphertext: EGNLHT.

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10. Let $S = \{(0, 1, 1), (-2, 2, 0), (0, 0, 1)\}$.

(a) Show S is a basis.

(b) Orthogonalize it.

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11. Prove the following:

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ be a linearly dependent subset of the vector space V. Say

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n = 0$$

where a_1 is not zero.

Show that one of the elements of S is a lenear combination of the other elements of S.

12. Asssume $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^n$ satisfy the following properties

- $\mathbf{u}_1 \cdot \mathbf{u}_1 = 1$, $\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$ and $\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$
- $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_3 \cdot \mathbf{u}_2 = 0$

Show the change of basis matrix $P_{\text{STANDARD} \to B}$ is given by the matrix with rows $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.