Math 6250: Test 2 Review

1 Limits of Functions

1. Let I be some interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$ and $gI \to \mathbb{R}$ and let $c \in I$. Assume $\lim_{x\to c} f(x) = F$ and $\lim_{x\to c} g(x) = G$ and that $k \in \mathbb{R}$. Prove the following:

   (a) $\lim_{x\to c} kf(x) = kF$

   (b) $\lim_{x\to c} f(x) + g(x) = F + G$

   (c) $\lim_{x\to c} f(x)g(x) = FG$

2. Prove the following.

   (a) $\lim_{x\to -2} 3x - 1 = -7$

   (b) $\lim_{x\to -2} x^2 = 4$

   (c) $\lim_{x\to 4} x^2 + 2x$

   (d) $\lim_{x\to 4} \frac{1}{x} = \frac{1}{4}$

   (e) $\lim_{x\to 9} \sqrt{x} = 3$

3. Show the following inequalities (for $x \in (0, \pi/2])$ using graphical techniques:

   - $\sin(x) \leq x$
   - $x \leq \tan(x)$
   - $\cos(\theta) \leq \frac{\sin(\theta}{\theta} \leq 1$

4. Use the squeeze Theorem to show

   (a) $\lim\limits_{x\to 0} x \sin(1/x^2) = 0$

   (b) $\lim\limits_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$ Use Problem 3

5. Use the fact that

$$\lim_{a\to 0} \frac{\sin(a)}{a} = 1$$

   to solve the following (do not use $\varepsilon - \delta$):

   (a) $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

(b) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

(c) $\lim_{x\to 0} \frac{\sin^2(x)}{x^2}$

(d) $\lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$. Hint us Problem 19e

2 Continuity

6. Prove the following (use $\varepsilon - \delta$ definition).

   (a) $f(x) = x^2$ is continuous at $x = -2$.

   (b) $f(x) = x^2$ is continuous.

   (c) $f(x) = \sqrt{x}$ is continuous at $x = 0$.

   (d) $f(x) = \sqrt{x}$ is continuous at $x = 4$.

   (e) $f(x) = \sqrt{x}$ is continuous.

   (f) $f(x) = \frac{1}{x}$ is continuous at $x = 3$.

   (g) $f(x) = \frac{1}{x}$ is continuous.

7. For the function below (you do not need the $\varepsilon - \delta$ definition here). Is it continuous? Is it differentiable? If yes prove. If no disprove.

$$f(x) = \begin{cases} \frac{x^4-2x^2+1}{|x|-1} & : x \neq 1 \text{ and } x \neq -1 \\ 0 & : x = 1, -1 \end{cases}$$

8. For each the functions below. Is it continuous (you do not need the $\varepsilon - \delta$ definition here)? Is it differentiable? If yes prove. If no disprove. Also graph function.

   (a) $f(x) = \begin{cases} \sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

   (b) $f(x) = \begin{cases} x\sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

   (c) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

9. Show the following are continuous (or not). Do not use $\varepsilon - \delta$.

   (a) $f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$ at $x = 0$.

(b) $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

(c) $f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

(d) $f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

3 Derivatives

10. Prove the following (you do not need $\varepsilon - \delta$)

    (a) $[kf]'(x) = kf'(x)$

    (b) $[f+g]'(x) = f'(x) + g'(x)$

    (c) $[fg]'(x) = f'(x)g(x) + f(x)g'(x)$

    (d) $[1/f]'(x) = \frac{-f'(x)}{(f(x))^2}$

    (e) $[f/g]'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$. Use Problem 10c and Problem 10d to show this.

11. Prove
    if f is differentiable at $x = c$ then f is continuous at $x = c$.

12. Compute the following derivatives at the indicated point(s).

    (a) $f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$ at $x = 0$.

    (b) $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

    (c) $f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

    (d) $f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$.

    (e) $f(x) = \begin{cases} \frac{|x|-4x}{x} & x \neq 0 \\ -4 & x = 0 \end{cases}$ at $x = 0$.

    (f) $f(x) = \begin{cases} \frac{|x^2|-4x}{x} & x \neq 0 \\ -4 & x = 0 \end{cases}$ at $x = 0$.

(g) $f(x) = \begin{cases} \frac{|x^3|-4x}{x} & x \neq 0 \\ -4 & x = 0 \end{cases}$ at $x = 0$.

(h) Show $[\sin(x)]' = \cos(x)$. Use Problem 5d to help.

(i) Use $[\sin(x)]' = \cos(x)$ to compute the derivatives for $\cos(x)$, $\tan(x)$, $\cot(x)$, $\sec(x)$ and $\csc(x)$.

13. State Rolle's Theorem.

14. State the Mean Value Theorem (MVT).

15. Use Rolle's Theorem to prove the MVT.

16. What does the MVT tell us about the following functions over the given interval. Find c as in the MVT.

    (a) $f(x) = x^2$ over the interval $[-1, 3]$

    (b) $f(x) = |x|^3$ over the interval $[-1, 3]$

    (c) $f(x) = |x|$ over the interval $[-1, 3]$

4 Trigonometry and Dimensions

17. Compute the Taylor series for the following functions at $c = 0$.

    (a) $f(x) = (x - 3)^4$

    (b) $f(x) = \sin(2x)$

    (c) $f(x) = e^{5x}$

18. Prove $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

19. Prove the following trigonometric identities:

    (a) $\sin(2x) = 2\sin(x)\cos(x)$

    (b) $\cos(2x) = \cos^2(x) - \sin^2(x)$

    (c) $\sin(4x) = 4\sin(x)\cos^3(x) - 4\sin^3(x)\cos(x)$

    (d) $\cos(4x) = \cos^4(x) - 2\sin^2(x)\cos^2(x) + \sin^4(x)$

    (e) $\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$

    (f) $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

20. Compute the dimension of a fractal as in class.