Name: ______________________________

1 Basic Proofs

1. Complete the following:

   (a) Disprove the statement: Let $n \in \mathbb{Z}$. If $n^3 + n^2 + n + 1$ is odd then n is odd.

   (b) Prove the statement: Let $n \in \mathbb{Z}$. If $n^3 + n^2 + n + 1$ is even then n is odd.

2. Let x be an odd integer. Show $x^2 - 1$ is divisible by 4.

3. Let $n \in \mathbb{Z}$. Show $(n + 2)^2 - (n)^2$ is divisible by 4. Here I used four cases, the four possible remainders when dividing n by 4.

   case 1. n = 4q, case 2. n = 4q+1, case 3. n = 4q+2, and case 4. n = 4q+3,

4. Let $x \in Q \setminus \{0\}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$. Show $xy \in \mathbb{R} \setminus \mathbb{Q}$.

5. Let $x, y \in Q$. Show $xy \in \mathbb{Q}$.

6. Disprove the following:
   If $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$ then $xy \in \mathbb{R} \setminus \mathbb{Q}$.

2 Induction

7. Which of the following are well ordered. No proof is needed.

   (a) $A = \mathbb{N}$

   (b) $B = \mathbb{Z}$

   (c) The set of all even integers.

   (d) $D = \{q \in \mathbb{Q} | q > 0\}$

8. Prove. Let $B \subset A$. If A is a well ordered set of numbers than B is a well ordered set of numbers.

9. Let $n \in \mathbb{N}$. Show $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

10. Let $n \in \mathbb{Z}$ and $n > 4$. Show $n! > 2^n$.

11. Let $n \in \mathbb{N}$. Show $4|5^n - 1$.

12. Prove $8|5^{2n} - 1$ for all $n \in \mathbb{N}$.

13. Prove $n! > 2^n$ for all $n \in \mathbb{N}$ with $n > 3$.

14. Define the sequence
$$a_1 = 1 \text{ and } a_n = \sqrt{a_{n-1} + 3}$$
   (a) Prove a_n is increasing. That is, prove $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.
   (b) Prove a_n is bounded above by 6. That is, prove $a_n \leq 6$ for all $n \in \mathbb{N}$.

3 Relations

15. Let R, a relation on $A = \{1, 2, 3\}$, be defined as follows
$$\mathcal{R} = \{(1,2), (2,3), (3,1), (1,1)\}$$
   (a) Compute $dom(\mathcal{R})$, $range(\mathcal{R})$, and $\mathcal{R}^{-1}$.
   (b) Is $\mathcal{R}$ reflexive symmetric or transitive? Show this.

16. Let R be a relation on $\mathbb{Z}$ be defined as follows
$$a\mathcal{R}b \iff 4|a + 3b.$$
   (a) Write out seven element of this relation.
   (b) Note that $\mathcal{R}$ is reflexive, symmetric and transitive? Prove.

17. For the following relation defined on $\mathbb{Z}$
$$a\mathcal{R}b \iff 5|a - b$$
prove that $\mathcal{R}$ is an equivalence relation.

18. Using the equivalence relation above, prove that addition over $\mathcal{R}$ is well defined. That is show

if $a_1\mathcal{R}b_1$ and $a_2\mathcal{R}b_2$ then $a_1 + a_2\mathcal{R}b_1 + b_2$.

19. Solve the following problems for x in $\mathbb{Z}_n$ where $x \in \{0, 1, 2, \ldots, n-1\}$.
   (a) $[x] + [3] = [11]$ in $\mathbb{Z}_3$
   (b) $[3][100] = [x]$ in $\mathbb{Z}_7$
   (c) $[3][x] = [1]$ in $\mathbb{Z}_7$
   (d) $[3][x] = [100]$ in $\mathbb{Z}_7$
   (e) $[2][x] = [4]$ in $\mathbb{Z}_8$
   (f) $[4][x] = [2]$ in $\mathbb{Z}_8$

20. For the relations defined below what are the equivalence classes?
    (a) $a\mathcal{R}b \iff 3|a+2b$ on $\mathbb{Z}$
    (b) $\mathcal{R} = \{(1,1),(2,2),(3,3),(1,2),(2,1))\}$ on $S = \{1,2,3\}$.
21. $P_1 = \{1,2\}$, $P_2 = \{3\}$, and $P_3 = \{4\}$ is a partition of the set $S = \{1,2,3,4\}$. What is the corresponding equivalence relation on S?

4 Functions

22. Define $f : \{1,2,3\} \to \{4,7,9\}$ by $f(1) = 4$, $f(2) = 4$ and $f(3) = 9$.
    (a) Is f injective, surjective or bijective? Compute
    (b) Compute $f(\{1,2\})$, $f^{-1}(\{1,4\})$ and $f \circ f^{-1}(\{4,9\})$.
23. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n-1$. Is f injective, surjective or bijective? Prove or disprove.
24. Solve the following from your book 10.1, 10.2, 10.11
25. Prove the following for the functions $f : A \to B$ and $g : B \to C$.
    (a) If f is injective and g is injective then $g \circ f$ is injective.
    (b) If f is surjective and g is surjective then $g \circ f$ is surjective.
    (c) If $g \circ f$ is injective then f is injective.
    (d) If $g \circ f$ is surjective then g is surjective.
26. Let $f : [1,3] \to [8,14]$ be defined by $f(x) = 3x+5$.
    (a) Prove f is a bijection.
    (b) Find the inverse of f.
27. Let $f : (0,\infty) \to (0,1)$ be defined by $f(x) = \frac{x}{1+x}$.
    (a) Prove f is a bijection.
    (b) Find the inverse of f.
28. Define $f : (-\infty,0) \to [0,\infty)$ by $f(x) = x^2$.
    (a) Is f injective, surjective or bijective? Prove or disprove.
    (b) Compute $f((-2,2))$, $f^{-1}((-2,2))$, $f^{-1} \circ f(\{4,9\})$ and $f \circ f^{-1}(\{4,9\})$.
29. Define $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}$ by $f(x) = \frac{x}{x-2}$. Is f injective, surjective or bijective? Prove or disprove.
30. Find all bijective functions from $\{1,2,3\}$ to $\{1,2,3\}$. How many functions did you come up with?

5 Cardinality

31. What is the definition of cardinality, infinite, finite, countable and uncountable.

32. Show the following:

    (a) $A \sim \mathbb{N}$ where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.

    (b) $\mathbb{Z} \sim \mathbb{N}$

    (c) $(1, 2] \sim [1, 5)$

    (d) $(0, \infty) \sim (0, 1)$