Name: ____________________

Complete Test Review 1 and 2 and here are a few new problems covering the new topics.

1 Groups

1. Let G be a group and assume $g^2 = e$ for all $g \in G$. Prove G is Abelian.
2. Let G be a group and let $a \in G$. Prove $(a^{-1})^{-1} = a$.
3. Let G be a group and let $a, b \in G$. Prove $(ab)^{-1} = b^{-1}a^{-1}$.

2 Subgroups

Theorem [the subgroup test] Let $(G, *)$ be a group and H be a nonempty subset of G. If

- for all $a, b \in H$ $a * b \in H$, and
- for all $a \in H$ $a^{-1} \in H$

then H is a subgroup.

4. Note that $(\mathbb{Z}, +)$ is a group and

$$E = \{2n | n \in \mathbb{Z}\}$$

is a subset of $\mathbb{Z}$. Show E is a subgroup of $\mathbb{Z}$.

5. Let G be a group and let a be a fixed element in G. Define

$$H = \{axa^{-1} | x \in G\}.$$

Note H is a subset of G. Prove H is a subgroup.

3 Isomorphisms

6. Write down the definition of isomorphism.
7. Define an isomorphism from $(\mathbb{Z}_4, +)$ to $(\mathbb{Z}_8^*, \cdot)$ or show that no such isomorphism exists.
8. Define an isomorphism from $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ to $(\mathbb{Z}_8^*, \cdot)$ or show that no such isomorphism exists.
9. Define an isomorphism from $(\mathbb{Z}_4, +)$ to $(\mathbb{Z}_8^*, \cdot)$ or show that no such isomorphism exists.

10. For the function $f : \mathbb{Z} \to \mathbb{Z}_5$ defined by

$$f(n) = n \mod 5.$$

Show $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{Z}_5$. Is f an isomorphism? Why or why not?

11. Note that $(\mathbb{Z}, +)$ is a group and

$$E = \{2n | n \in \mathbb{Z}\}$$

is a subgroup of $\mathbb{Z}$. Define an isomorphism from $\mathbb{Z}$ to E. Prove it is an isomorphism.

12. For the group $(S_3, \circ)$ Show

$$H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \right\}$$

is a group. Show $(Z_4, +)$ is isomorphic to $(H, \circ)$.

13. Show $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic with the following isomorphism $f : \mathbb{R} \to \mathbb{R}^+$

$$f(x) = e^x.$$

That is show f is a bijection and show that f satisfies

$$f(a+b) = f(a) \cdot f(b) \text{ for all } a, b \in \mathbb{R}.$$

14. Let G_1 and G_2 be groups and let $f : G_1 \to G_2$ be an isomorphism. Let $a \in G_1$.
    (a) What group is $f(a)$ in?
    (b) Prove if $o(a) = n$ then the $o(f(a)) = n$.

15. Let G_1 and G_2 be groups and let $f : G_1 \to G_2$ be an isomorphism. Let $a \in G_1$.
    (a) Prove $f(a)^{-1} = f(a^{-1})$.