Name: ______________________________

1. State the Peano Axioms. The set $\mathbb{N}$ is the only set which satisfies these axioms.

   (a) Which Axiom fails for $\mathbb{Z}$?

   (b) Which Axiom fails for $A = [0, \infty)$ a subset of $\mathbb{R}$?

2. Write the definition of the equivalence classes that we used to define the integers and rationals.

3. What is the axiom needed for the existence of the reals? State the name of the axiom and the axiom.

4. Suppose that α is an upper bound for a set S and that $\alpha \in S$. Then $\alpha = \sup S$.

5. Suppose that α is an supremum for a nonempty set S and that $\alpha \notin S$. Then S is infinite.

6. State the definition of countable. Prove

$$A = \mathbb{N} \times \mathbb{N}$$

is countable from the definition.

7. Define functions from $A = \{1, 2, 3\}$ to any other set you wish with the following properties:

   (a) f is not injective is onto.

   (b) f is injective but not onto.

   (c) f is not injective and not onto.

8. Define functions from $\mathbb{R}$ to $\mathbb{R}$ with the following properties:

   (a) f is not injective is onto.

   (b) f is injective but not onto.

   (c) f is not injective and not onto.

9. Show the sequence below converges by using the monotone convergence theorem.

$$a_1 = \frac{1}{1}, a_2 = \frac{1}{1} + \frac{1}{4}, a_2 = \frac{1}{1} + \frac{1}{4} + \frac{1}{9}, \ldots, a_n = \sum_{k=1}^{n} \frac{1}{k^2}, \ldots$$

You must show the sequence is bounded above. To do this mimic the technique we used to show the sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k}$$

diverged.

10. Prove $\lim_{n\to\infty} \frac{4n^2+1}{2n^2+n+1} = 2$

11. Prove $\lim_{x\to 3} x^3 - 5x = 12$

12. Prove
    If $\lim a_n = a$ then the $\lim a_n^2 = a^2$ using the $\epsilon - N$ definition for limit.

13. Find all complex solutions to $x^6 = i\frac{\sqrt{3}}{2}$

14. Graph the the function

$$h(x) = \begin{cases} x\sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

Prove $\lim_{x\to 0} h(x) = 0$ using the $\epsilon - \delta$ definition.

15. Write down the definitions

- $f : A \to B$ is **continuous at** $x = c$

- $f : A \to B$ is **continuous**

- $f : A \to B$ is **uniformly continuous**

16. Prove the following using the $\epsilon - \delta$ definition.

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is continuous at $x = -9$

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is continuous

17. Prove the following using the $\epsilon - \delta$ definition.

- $f : [-22, 10] \to \mathbb{R}$ where $f(x) = x^2$ is uniformly continuous

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not uniformly continuous

18. Define a sequence by the following

$$a_1 = \sqrt{2}, a_2 = \sqrt{2}^{\sqrt{2}}, a_3 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, a_4 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}}, \ldots$$

(a) Prove (a_n) is convergent and find its limit.

(b) Let $x \in \mathbb{R}^+$. Now define a sequence as

$$b_1 = x, b_2 = x^x, b_3 = x^{x^x}, b_4 = x^{x^{x^x}}, \ldots$$

Clearly, the sequence (b_n) converges if $x = \sqrt{2}$. For what values of x does (b_n) converge?