Math 6250: Final Review

1 Derivaties

1. Compuite the indicated derivative using the definition of the derivative.

   (a) $f(x) = \sqrt{x}$ at x

   (b) $f(x) = \frac{1}{x}$ at x

   (c) $f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ at $x = 0$

   (d) $f(x) = \begin{cases} \frac{|x|x^3 - x^2}{x} + 2x & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0$

2. Find $f'(x)$ for $f(x) = \sum_{n=0}^{\infty} (1/2)^n \cos(16^n \pi x)$.

3. Prove if f is differentiable at $x = c$ then is continuous at $x = c$.

2 Taylor Series

4. Compute the following Taylor series from the formula at the indicated point.

   (a) $f(x) = \sin(3x)$ at $c = 0$

   (b) $f(x) = \sin(3x)$ at $c = \pi/2$

   (c) $f(x) = x^2 - 3$ at $c = 1$

   (d) $f(x) = e^{x/2}$ at $c = 1$

5. Compute the following Taylor series at $c = 0$ from know series using algebra and calculus.

   (a) $f(x) = \frac{\sin(x)}{x}$.

   (b) $f(x) = \ln(1 + x)$

   (c) $f(x) = \arctan(x)$

   (d) $f(x) = \frac{\cos(x^2) - 1}{x^2}$

   (e) $f(x) = e^{x^2}$

6. Use the results from Problem 5 to find

   (a) a series representation of π

   (b) The sum of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$

   (c) a series representation of $e^{1/2}$

   (d) a series representation of $\int_{-1}^{1} e^{-x^2}\, dx$

3 The Basel Problem

4 Means

7. State the definition of the Arithmetic (AM), Geometric (GM) and Harmonic (HM) means. How are the three means related?

8. What is the Cauchy inequality? Use it to help you prove the inequality from Problem 7.

9. Using the inequality from Problem 7 solve the following.

   (a) Prove that if the product of n positive real numbers is 1, then their sum is at least n.

   (b) Let $(a_k)_{k=1}^n$ be a sequence of possitive reals. And let $(b_k)_{k=1}^n$ be a permutation of $(a_k)_{k=1}^n$. Then

   $$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n} \geq n$$

   (c) Show that $(1 + a^2)(1 + b^2) \geq 4ab$ for all $a, b \in \mathbb{R}$.

   (d) Let $x, y, z \in \mathbb{R}^+$ so that $xyz = 32$. Show

   $$x^2 + 4xy + 4y^2 + 2z^2 \geq 96.$$

   (e) When I drive to school in the morning I take the LIE for the first half and drive at 60 mph and take Route 25 for the second half of the trip and drive at 30 mph. What is my average speed? Note I am confused by what "half" means in this problem. Does it mean for half of the time or half of the distance. Compute both answers.

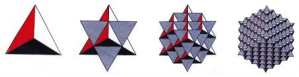
5 The Cantor Set

6 Fractional Dimensions

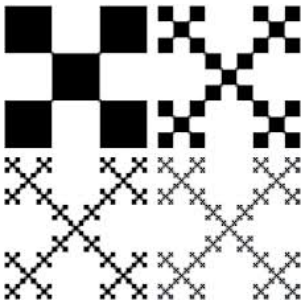
10. Compute the dimension of the following fractals.

(a)

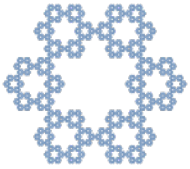
(b)



(c)

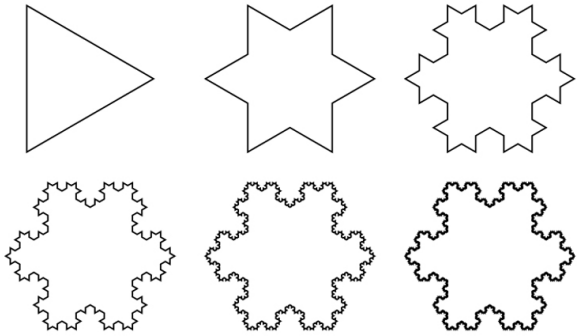


(d)



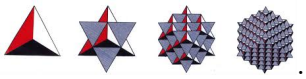
(e)

11. Back to our snowflake. Can you compute the perimeter of the snowflake?



Assuming the first image has sides of length 1 then I get the first image has 3 sides of length 1, so perimeter of 3 units. And the second image has edge lengths of

$\frac{1}{3}$ and has 12 edges; so the perimeter for the second shape is 12 times $\frac{1}{3}$ which is $\frac{12}{3}$.

12. Compute the surface area of  .

13. Design your own fractal.