Math 6250 Quiz 2

Name:____

1. Show $f : (0,1) \to (1,\infty)$ defined by $f(x) = \frac{1}{x}$ is bijective.

2. For the following find a bijective function from one set to the other (except for 2c). Demonstrate it is bijective.

   (a) $\mathbb{N} \sim \mathbb{Z}$

   (b) $\mathbb{N} \sim \mathbb{Q}$

   (c) $\mathbb{N} \not\sim \mathbb{R}$

   (d) $(0,1) \sim [0,1)$. I think this one is tricky.

3. Show if $f : A \to B$ is bijective and $g : B \to C$ is bijective then $g \circ f : A \to C$ is bijective.

4. Let A, B be nonempty bounded subsets of $\mathbb{R}$. Define $A+B = \{a+b | a \in A, b \in B\}$ $-A = \{-a | a \in A\}$

   (a) For $A = (1,3)$ and $B = [-4,-1]$. Compute $A + B$ and $-A$.

   (b) For $A = (1,3)$ and $B = [-4,-1]$. Compute $\sup(A)$, $\sup(B)$, $\sup(A+B)$ and $\sup(-A)$.

   (c) Prove the following fact.
   For any A, B nonempty bounded subsets of $\mathbb{R}$ we have that

   $$\sup(A) + \sup(B) = \sup(A+B).$$

   (d) Guess a similar fact about the $\sup(-A)$.

5. Prove the triangle inequality. That is for all $x, y \in \mathbb{R}$

   $$|x+y| \leq |x| + |y|.$$

   Hint: It is easier to show $|x+y|^2 \leq (|x|+|y|)^2$ by looking at various cases.

6. How have we defined the reals? The reals are also the only complete ordered field. What are the definitions for 1. complete. 2. ordered and 3. field. Look these up.

7. Prove the sup of set is unique.

8. Calculate the square roots of i.

9. Calculate the cube roots of $1 + i$

10. Use DeMoivre's to show: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

11. Show the following are monotone or not. State whether they are monotone increasing, monotone decreasing or not monotone. And prove it.

    (a) $a_n = \frac{1}{n}$.
    (b) Defined recursively as $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$.
    (c) Defined recursively as $a_1 = 1$ and $a_{n+1} = 1 + \frac{a_n}{a_n+1}$.
    (d) $a_n = \frac{-1}{n}$.
    (e) What do the sequences from 11b and 11c have to do with a well known sequence?

12. ~~Prove the Monotone convergence Theorem: That is~~
    ~~If (a_n) is a bounded and monotone sequence then (a_n) converges.~~

13. Use the Monotone Convergence Theorem to show that: the sequence defined as $a_1 = 1$ and $a_{n+1} = 1 + \frac{a_n}{a_n+1}$ converges.

14. Prove with $\varepsilon - N$ proof that $a_n = \frac{2n+1}{3n+5}$ converges.

15. Prove with $\varepsilon - N$ proof that the sequence defined below is not convergent.

$$a_n = \sum_{j=1}^{n} \frac{1}{j}$$

So $a_1 = \sum_{j=1}^{1} \frac{1}{j} = \frac{1}{1} = 1$, $a_2 = \sum_{j=1}^{2} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$ and $a_3 = \sum_{j=1}^{3} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$.