1 Expected Value

Let X be a RV with PF $f(x)$ and let u be a function of x. Then the **expected value** of $u(X)$ is defined as

$$\begin{aligned} E(u(X)) &= \sum u(X)f(x) \text{ for discrete RV} \\ &= \int u(X)f(x)dx \text{ for continuous RV} \\ &= \int_A u(X)f(x)dx + \sum_{x=x_i} u(X)f(x) \text{ for mixed RV} \end{aligned}$$

Problem 1.1. *Let X be a discrete RV with PF defined below.*

x	-1	2	3
$f(x)$	1/2	1/3	1/6

Compute

1. $E(X)$
2. $E(X^2)$
3. $E(e^{tX})$

Problem 1.2. *Let X be a continuous RV with PF defined below.*

$$f(x) = \frac{1}{4}e^{-x/4} \text{ for } x \geq 0$$

Compute

1. $E(X)$
2. $E(X^2)$
3. $E(e^{tX})$

Problem 1.3. *Let X be a mixed RV with PF defined below.*

$$f(x) = \begin{cases} 0 & x < 0 \\ x/6 & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

Compute

1. $E(X)$
2. $E(X^2)$
3. $E(e^{tX})$

2 Moments, Means and Other Statistics

Let X be a RV. The n^{th} **moment** of X is defined as $E(X^n)$ and the n^{th} **central moment** of X is defined as $E\left((X - E(X))^n\right)$.

So the 0^{th} moment would be $E(X^0) = E(1) = 1$.

So the 1^{st} moment would be $E(X)$. We call this moment the **mean** of X and write as $\mu_X = E(X)$.

And the 2^{nd} central moment would be $E((X - E(X))^2)$. We call this moment the **variance** of X and write as $VAR(X) = \sigma_X^2$.

Two other common statistics are the **standard deviation**

$$SD(X) = \sqrt{\sigma_X^2}$$

and the **Coefficient of variance**

$$CV(X) = \frac{\sigma_X}{E(X)}.$$

Problem 2.1. *Let X be a discrete RV with PF defined below.*

x	-1	2	3
$f(x)$	1/2	1/3	1/6

Compute μ_X, $VAR(X)$, $SD(X)$ and $CV(X)$.

Problem 2.2. *Let X be a continuous RV with PF defined below.*

$$f(x) = \frac{1}{4}e^{-x/4} \text{ for } x \geq 0$$

Compute μ_X, $VAR(X)$, $SD(X)$ and $CV(X)$.

3 Linearity of the Expected Value

Let X be a random variable, let u and v be functions of x and let $a, b \in \mathbb{R}$ then

$$E(au(X) + bv(X)) = aE(u(X)) + bE(v(X))$$

Also note

1. $E(aX) = aE(X)$
2. $VAR(X) = E(X^2) - E(X)^2$
3. $VAR(aX) = a^2 VAR(X)$

Prove Item 2 and Item 3.

Problem 3.1. *Two challenging problems.*

1. *Let X be a constant RV. (ie $Pr(X = c) = 1$ for some $c \in \mathbb{R}$). Prove that $E(X) = c$ and that $VAR(X) = 0$.*
2. *Prove if $VAR(X) = 0$ then X is a constant RV.*

4 Moment Generating Function

The **Moment Generating Function** of a RV X is $M(t) = E(e^{tX})$.

Also note

1. $M(0) = 1$
2. $\frac{d^n}{dt^n}[M(t)]_{t=0} = E(X^n)$
3. $M'(0) = E(X)$
4. $VAR(aX) = M''(0) - [M'(0)]^2$

Prove Item 3 and Item 4.

Problem 4.1. *Find the MGF for*

1. $Pr(X = 0) = 1/2$, $Pr(X = 1) = 1/3$ *and* $Pr(X = 2) = 1/6$.
2. $f(n) = \left(\frac{1}{2}\right)^{n+1}$ *for* $n = 0, 1, 2, 3, \ldots$.
3. $f(x) = \frac{1}{b-a}$ *for* $a < x < b$.
4. $f(x) = 7e^{-7x}$ *for* $x > 0$.

Problem 4.2. *Let* $M(t) = (1 - 2500t)^{-4}$ *be the MGF for some RV X. Compute E(X), VAR(X) and CV(X)*

5 Optional Problem

Let X be a RV that only assumes nonnegative values and let $m \in \mathbb{R}^+$. Let $Y = min(X, m)$. Then

$$E(Y) = \int_0^m (1 - F(x))\, dx$$

where $F(X)$ is the CDF for X.

Sketch.

$$
\begin{aligned}
E(Y) &= \int yf(x)\,dx \\
&= \int_0^\infty min(x,m)f(x)\,dx \\
&= \int_0^m min(x,m)f(x)\,dx + \int_m^\infty min(x,m)f(x)\,dx \\
&= \int_0^m xf(x)\,dx + \int_m^\infty mf(x)\,dx \\
&= \int_0^m x[1-F(x)]'\,dx + \int_m^\infty mf(x)\,dx \text{ since F'(x) = f(x)} \\
&= x[1-F(x)]_0^m + \int_0^m [1-F(x)]\,dx + \int_m^\infty mf(x)\,dx \text{ by IPB} \\
&= -m[1-F(m)] + \int_0^m [1-F(x)]\,dx + m\int_m^\infty f(x)\,dx \\
&= -m[1-Pr(X<m)] + \int_0^m [1-F(x)]\,dx + mPr(X>m) \\
&= -mPr(X>m) + \int_0^m [1-F(x)]\,dx + mPr(X>m) \\
&= \int_0^m [1-F(x)]\,dx
\end{aligned}
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