1 Probability Space

Let S be a set and let $\mathcal{A}$[1] be a collection of subsets of S. We call the elements of $\mathcal{A}$ events. We say S is a **probability space** if there is a function Pr from $\mathcal{A}$ to $[0,1]$ so that

1. $Pr(S) = 1$,
2. $Pr(\emptyset) = 0$, and
3. If $A \cup B = \emptyset$ then $Pr(A \cap B) = Pr(A) + Pr(B)$. Moreover for a countable collection of pairwise disjoint sets (A_n) we have

$$Pr(\cup A_n) = \sum Pr(A_n).$$

The following statements for probability space S and the given events.

1. Let A, B be events. If $A \subseteq B$ then $Pr(A) \leq Pr(B)$
2. Let A, B be events. Then $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
3. For any event E, $Pr(E^c) = 1 - Pr(E)$.
4. For any event E, A, $Pr(A) = Pr(A \cap E) + Pr(A \cap E^c)$.
5. Let $E_1, E_2, \cdots, E_n$ form a partition for the set S. Then

$$Pr(A) = Pr(A \cap E_1) + Pr(A \cap E_2) + \cdots + Pr(A \cap E_n)$$

Problem 1.1. *Flip a coin 3 times. Write out S.*

1. *Assuming the coin is fair what are the probabilities of each event?*
2. *Assuming the coin is biased, say the probability of heads on a single flip is* $1/3$, *then what is the probability of each event?*

2 Conditional Probability

- If $A \cap B = \emptyset$ then $Pr(A \cup B) = Pr(A) + Pr(B)$
- $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$
- $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$
- $Pr(A) = Pr(A \cap B) + Pr(A \cap B^c) = Pr(A|B)Pr(B) + Pr(A|B^c)Pr(B^c)$

[1]Note the collection $\mathcal{A}$ must be a sigma algebra but that is beyond the scope of this course.

- Law of Total Probability. If E_1, E_2, E_3 is a partition of S then

$$\begin{aligned} Pr(A) &= Pr(A \cap E_1) + Pr(A \cap E_2) + Pr(A \cap E_3) \\ &= Pr(A|E_1)Pr(E_1) + Pr(A|E_2)Pr(E_1) + Pr(A|E_3)Pr(E_3) \end{aligned}$$

- Bayes Rule for two sets

$$\begin{aligned} Pr(B|A) = \frac{Pr(B \cap A)}{Pr(B)} &= \frac{Pr(B \cap A)}{Pr(A \cap B) + Pr(A \cap B^c)} \\ &= \frac{Pr(A|B)Pr(B)}{Pr(A|B)Pr(B) + Pr(A|B^c)Pr(B^c)} \end{aligned}$$

- Bayes Rule for three sets

$$Pr(E_1|A) = \frac{Pr(A|E_1)Pr(E_1)}{Pr(A|E_1)Pr(E_1) + Pr(A|E_2)Pr(E_2) + Pr(A|E_3)Pr(E_3)}$$

Problem 2.1. *Roll a fair four sided die twice and record the two rolls. Let event*

- *A be a sum of 5*
- *B be a sum of 6*
- *C a roll of 3 on the first roll*

1. *Write down S.*
2. *What are the probabilities of A, B and C?*
3. *What is the probability of A given C?*
4. *What is the probability of B given C?*
5. *Are the events A and C independent?*
6. *Are the events C and B independent?*

Problem 2.2. *We have an urn with 4 red balls, eleven green balls. We will select two without replacement.*

1. *What is the probability that we draw a red ball the first draw?*
2. *What is the probability that we draw a green ball the first draw?*

3. *What is the probability that we draw a red ball on the second draw given that the first draw was a red ball?*

4. *What is the probability that we draw a red ball on the second draw given that the first draw was a green ball?*

5. *What is the probability that we draw a red ball on the second draw (use the law of total probability here)?*

6. *What is the probability that we draw a red ball on the first draw given that we drew a red ball on the first draw (use Bayes' rule here)?*

Problem 2.3. *We have an urn with 4 red balls, two green balls, three blue balls and 6 purple balls. We will select two without replacement.*

1. *What is the probability that we draw a red ball on the second draw given that the first draw was a red ball?*

2. *What is the probability that we draw a red ball on the second draw given that the first draw was a green ball?*

3. *What is the probability that we draw a red ball on the second draw given that the first draw was a blue ball?*

4. *What is the probability that we draw a red ball on the second draw given that the first draw was a purple ball?*

5. *What is the probability that we draw a red ball on the second draw (use the law of total probability here)?*

6. *What is the probability that we draw a red ball on the first draw given that we drew a red ball on the first draw (use Bayes' rule here)?*

Problem 2.4. *Look at the following table for the outcome of an experimental drug.*

	Drug A success	Drug A Failure	Drug B Success	Drug B Failure
Male	23	300	25	250
Female	225	800	73	231

Read the table as 23 men took Drug A and the drug was successful and 300 men took Drug A and it was a failure. So 23 out of 323 is the success rate for Drug A for men.

1. *How many people took the drug survey, male and female combined for Drug A? How many had a successful experience with Drug A? What is the success rate for Drug A?*

2. *What is the success rate (male and female combined) for Drug B?*

3. *Which Drug seems better overall?*

4. *Compute Probability of success for Drug A given that the participant was male.*

5. *Compute Probability of success for Drug A given that the participant was female.*
6. *Compute Probability of success for Drug B given that the participant was male.*
7. *Compute Probability of success for Drug B given that the participant was female.*
8. *Now which drug seems better?*
9. *Compute Probability that the participant was male given that the drug was successful (Use Bayes' Rule).*

2.1 Combinatorial Probability

- Permutations ${}_nP_k = \frac{n!}{(n-k)!}$
- Combinations ${}_nP_k = \binom{n}{k} = \frac{n!}{(n-k)!k!}$
- Binomial Theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$
- Binomial Distribution $Pr(x=k)\binom{n}{k}p^k q^{n-k}$ where p is the probability of success and q is the probability of failure.

3 Random Variables and Expected Value

Let $X : S \to \mathbb{R}$. We say X is a **random variable** if

$$A_a = \{s \in S | X(s) \leq a\}$$

has a probability for all $a \in \mathbb{R}$. Then $f(x)$ is called the probability function if it satisfies

- Discrete $Pr(X = a) = f_X(a)$
- Continuous $Pr(a \leq X \leq b) = \int_a^b f(x)dx$

Problem 3.1. *Flip a fair coin 3 times. Let X be the count of heads be a random variable.*

1. *What is S?*
2. *What values can X have?*
3. *Compute $f(x)$ the probability function.*
4. *Compute $Pr(X \leq 2)$.*

Problem 3.2. *Let $f(x) = e^{-x}$ for $x \geq 0$ be the probability function for some random variable X.*

1. *What is S?*

2. *What values can X have?*

3. *Compute* $Pr(X \leq 2)$.

4. *Compute* $Pr(X \geq 2)$.

5. *Compute* $Pr(X \geq 5|X \geq 3)$.

6. *Compare your answers to Problem 4 and Problem 5*

Problem 3.3. *Let* $f(x) = 2e^{-2x}$ *for* $x > 0$ *be the pdf of some RV X. Compute the following*

1. $Pr(2 < X < 3)$. .

2. $Pr(4 < X < 5|x > 2)$.

3. $Pr(11 < X < 12|x > 9)$.

4. *CDF of X,* $F(x)$.

5. *Use* $F(x)$ *to compute* $Pr(X < 3)$

6. *Use* $F(x)$ *to compute* $Pr(X < 2)$

7. *Use Problem 6 and Problem 5 to compute* $Pr(2 < X < 3)$. *Compare to Problem 1.*

8. *Compute* $E(x)$ *and* $Var(X)$.

Problem 3.4. *Let* $f(x) = \frac{5x^4}{32}$ *for* $0 < x < 2$ *be the pdf of some RV X. Compute the following*

1. $Pr(X < 0.5)$.

2. $Pr(X < 0.5|x < 0.7)$.

3. *CDF of X,* $F(x)$.

4. *Use* $F(x)$ *to compute* $Pr(X < 0.5)$

5. *Compute* $E(x)$ *and* $Var(X)$.

Problem 3.5. *Let* $F(x) = x^2$ *for* $0 < x < 1$ *be the CDF of a RV X. Compute the following*

1. *the pdf* $f(x)$.

2. $Pr(1/2 < X < 1/3)$.

3. *Compute* $E(x)$ *and* $Var(X)$.

Problem 3.6. *Let* $F(x) = x^2$ *for* $0 < x < 1$ *be the CDF of a RV X. Compute the following*

1. *the pdf* $f(x)$.

2. $Pr(1/2 < X < 1/3)$.

3. *Compute* $E(x)$ *and* $Var(X)$.