Math 6250: Test 1

Name:____

1. Define the follwing relation on $\mathbb{Z}$

   $$a\mathcal{R}b \iff 5|4a+b$$

   Prove $\mathcal{R}$ is reflexive and symmetric.

2. Let $f : A \to B$ and let $g : B \to C$. Prove if f and g are injective then $g \circ f$ is injective..

3. Let $A \subset \mathbb{R}$ that is nonempty and bounded above. Let $\alpha = \sup(A)$. Prove if $\alpha \notin A$ then A is infinite.

4. State the completeness axiom.

5. Prove limit of sequence $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

6. Prove **one** of the following:

   (a) $\mathbb{Q} \sim \mathbb{N}$

   (b) $\mathbb{R} \not\sim \mathbb{N}$

7. Prove **one** of the following:

   (a) If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n b_n = ab$.

   (b) The series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

8. Find the limit and prove your limit is correct (use the $\varepsilon - \delta$ definition for limits).

   $$\lim_{x\to -3} x^2 - 3x$$

9. Prove function $f(x) = \frac{1}{x}$ is continuous at the point $x = 7$ (use the $\varepsilon - \delta$ definition for continuity).

10. Let $p \in \mathbb{R}^+$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy

    $$|f(x) - f(y)| \leq |x-y|^p \text{ for all } x, y \in R.$$

    (a) Prove f is continuous (use the $\varepsilon - \delta$ definition for continuity).

    (b) Does $f : [0,\infty) \to \mathbb{R}$ where $f(x) = \sqrt{x}$ satisfy this property with $p = 1$? Prove or disprove.

ec 1. Find all complex solutions to $x^4 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

ec 2. Graph the the function $f(x) = \frac{1}{x}$ and the function $g(x) = \sin(x)$. Use these two to graph

$$h(x) = \begin{cases} x\sin(\frac{1}{x}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

Prove $h(x)$ continuous at the point $x = 0$ (use the $\varepsilon - \delta$ definition for continuity).