Math 4160 - Test 2 Review

1 Some Old Stuff

1. For the following subspaces W compute $W^{\perp}$.
   (a) $W = \{(x, y)|x + y = 0\}$
   (b) $W = \{(x, y, z)|x + y = 0\}$
   (c) $W = \{(x, y, z)|x + y = 0 \text{ and } 3y - z = 0\}$

2 Inner Products

2. Start with Quiz 5.

3. Prove $\langle u + v, u - v\rangle = \|u\|^2 - \|v\|^2$ in a vector space over $\mathbb{R}$.

4. Prove $\langle u, v\rangle = \frac{1}{4}\|u + v\|^2 - \frac{1}{4}\|u - v\|^2$ in a vector space over $\mathbb{R}$.

5. Let e_1, e_2, e_3 be an orthonormal basis for an inner product space V. Prove
$$\|a_1e_1 + a_2e_2 + a_3e_3\| = \sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2}.$$

6. Let $u, v \in V$ where V is a real inner product space. Prove u is othogonal to w where w is defined as
$$w = v - \frac{\langle u, v\rangle}{\|u\|^2}u$$

7. Assume $\|Tv\| \leq \|v\|$ for all $v \in V$ an inner product space. Show $T - \sqrt{2}I$ is invertible.

   Hint: Note λ is an eigenvalue iff $T - \lambda I$ is not invertible.

8. Let $u_1 = (1, 2, 3), u_2 = (1, 1, 3)$ and $u_3 = (0, -1, 2) \in \mathbb{R}^3$. Use Gram-Schmidt Orthoganilzation to find an o-n basis.

9. Let $p_1 = 1, p_2 = x$ and $p_3 = x^2 \in \mathcal{P}_2(\mathbb{R})$. Use Gram-Schmidt Orthoganilzation to find an o-n basis. Use the inner product below
$$\langle f, g\rangle = \int_0^1 fg dx.$$

10. Find $U^{\perp}$ for the following sets U

(a) $U = \{(1,0,1)\} \subseteq \mathbb{R}^3$ with the usual inner product.

(b) $U = \{1\} \subseteq \mathcal{P}_2(\mathbb{R})$ with the inner product below

$$\langle f, g \rangle = \int_0^1 fg dx.$$

11. Find eigenvalues and cooresponding eigenvectors for the following operators

(a) $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ where $T(p) = p'$

(b) $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ where $T(p) = xp'$

12. Let $T \in \mathcal{L}(V)$ and let $v, w \in V$ be nonzero vectors so that

$$T(u) = 3w \text{ and } T(w) = 3u.$$

Prove that either 3 or -3 is an eigenvalue of T.

13. Find the least squares solution of the equation $A\mathbf{x} = b$. Also find the least squares error vector and least squares error of the stated equation. Verify that the least squares error vector is orthogonal to the column space of A.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}$

14. Find the least squares straight line fit $y = ax + b$ to the data points, and show that the result is reasonable by graphing the fitted line and plotting the data in the same coordinate system.

(a) $(0,0)$, $(1,2)$, $(2,7)$

(b) $(0,1), (2,0), (3,1), (3,2)$

15. Find the least squares quadratic fit $y = a_0 + a_1 x + a_2 x^2$ to the data points, and show that the result is reasonable by graphing the fitted curve and plotting the data in the same coordinate system.

(a) $(2,0), (3,10), (5,48), (6,76)$

(b) $(1,2), (0,1), (1,0), (2,4)$

3 Diagonalizataion and Quadratic Forms

16. Show that the matrix is orthogonal three ways: first by calculating $A^T A$, then by using part (b) of Theorem 7.1.1, and then by using part (c) of Theorem 7.1.1 listed below

    THEOREM 7.1.1 The following are equivalent for an $n \times n$ matrix A.

    (a) A is orthogonal.

    (b) The row vectors of A form an orthonormal set in R n with the Euclidean inner product.

    (c) The column vectors of A form an orthonormal set in R n with the Euclidean inner product.

$$A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

17. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the orthogonal matrix in Problem 16. Find $T_A(\mathbf{x})$ for the vector $\mathbf{x} = (2, 3, 5)$, and confirm that $\|T_A(\mathbf{x})\| = \|x\|$ relative to the Euclidean inner product on $\mathbb{R}^3$.

18. What conditions must a and b satisfy for the matrix A to be orthogonal?

$$A = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$$

19. Under what conditions will a diagonal matrix be orthogonal?

20. Prove that if $\mathbf{x}$ is an $n \times 1$ matrix, then the matrix

$$A = I_n - \frac{2}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T$$

    is both orthogonal and symmetric.

21. Can an orthogonal operator $T_A : \mathbb{R}^n \to \mathbb{R}^n$ map nonzero vectors that are not orthogonal into orthogonal vectors? Justify your answer.

    Recall that A is orthogonal iff $Ax \cdot Ay = x \cdot y$ for all $x, y \in \mathbb{R}^n$

22. Find the spectral decomposition of the following matrices.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix},\ C = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

23. Prove that if A is any $m \times n$ matrix, then $A^T A$ has an orthonormal set of n eigenvectors.

    Recall. THEOREM 7.2.2 If A is a symmetric matrix with real entries, then:

    (a) The eigenvalues of A are all real numbers.
    (b) Eigenvectors from different eigenspaces are orthogonal.

3.1 Orthogonal Matrices and Orthogonal Diagonalization

24. Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables.

    (a) $Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$
    (b) $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$
    (c) $Q = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$
    (d) $Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

25. Identify the conic section represented by the equation.

    (a) $2x^2 + 5y^2 = 20$
    (b) $x^2 - y^2 - 8 = 0$
    (c) $4x^2 - 5y^2 = 20$
    (d) $x - 3 = -y^2$
    (e) $7y^2 - 2x = 0$
    (f) $x^2 + y^2 - 25 = 0$

26. Classify the matrix as positive definite, negative definite, or indefinite.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix},\ B = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

27. find all values of k for which the quadratic form is positive definite.

    (a) $5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$
    (b) $3x_1^2 + x_2^2 + 2x_3^2 - 2x_1x_3 + 2kx_2x_3$

3.2 Quadratic forms and Optimization

28. Subject to the constraint $x^2 + y^2 = 1$ determine the values of x and y at which the maximum and minimum occur.

    (a) $5x^2 - y^2$
    (b) $3x^2 + 7y2$
    (c) xy
    (d) $5x^2 + 5xy$

29. Find the maximum and minimum values of the given quadratic form subject to the constraint $x^2 + y^2 + z^2 = 1$ and determine the values of x , y , and z at which the maximum and minimum occur.

    (a) $9x^2 + 4y^2 + 3z^2$
    (b) $2x^2 + y^2 + z^2 + 2xy + 2xz$

30. find the maximum and minimum values of xy subject to the constraint $4x^2 + 8y^2 = 16$.

31. Find the maximum and minimum values of $x^2 + xy + 2y^2$ subject to the constraint $x^2 + 3y^2 = 16$.

3.3 Hermitian, Unitary and Normal

32. First fill in the missing entries to make the matrix Hermitian. Then verify that the eigenvalues of the Hermitian matrix A are real and that eigenvectors from different eigenspaces are orthogonal.

$$A = \begin{bmatrix} 3 & 2-3i \\ ???? & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & ???? \\ -2i & 2 \end{bmatrix}$$

33. Show that A is unitary, and find A^{-1}.

$$A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5}i \\ -\frac{4}{5} & \frac{3}{5}i \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$$

34. Find a unitary matrix P that diagonalizes the Hermitian matrix A, and determine P^*AP .

$$A = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -i \\ i & 3 \end{bmatrix}$$

35. A square complex matrix A is unitarily diagonalizable if and only if

$$AA^* = A^*A$$

Matrices with this property are said to be normal.

36. Let A be any $n \times n$ matrix with complex entries, and define the matrices B and C to be

$$B = \frac{1}{2}(A + A^*) \text{ and } C = \frac{1}{2i}(A - A^*)$$

(a) Show that B and C are Hermitian.

(b) Show that $A = B + iC$ and $A^* = B - iC$.

(c) What condition must B and C satisfy for A to be normal?

37. Show that

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ ie^{i\theta} & -ie^{-i\theta} \end{bmatrix}$$

is unitary for all real values of θ. Note that $e^{i\theta} = cos(\theta) + i\sin(\theta)$.

38. What relationship must exist between a matrix and its inverse if it is both Hermitian and unitary?