Math 4160 - Test 1 Review

1 Vector Spaces and Subspaces

1. Let $V = \mathbb{R}^3$ equipped with usual vector addition and scalar multiplication. Prove V is a vector space. That is, prove all 10 Axioms.

2. Let $V = \mathbb{R}^2$. And define the two operations

   $\oplus$: $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$
   $\odot$: $k \odot (x_1, y_1) = (kx_1, ky_1)$

   (a) Compute $(0, 4) \oplus (-2, 3)$ and compute $2 \odot (1, 1)$.
   (b) Show $\mathbf{0} \neq (0, 0)$.
   (c) Show $\mathbf{0} = (-1, -1)$.
   (d) Prove Axiom 5. That is, for each $\mathbf{v}$ find $-\mathbf{v}$ so that
   $$\mathbf{v} \oplus -\mathbf{v} = \mathbf{0}.$$
   (e) V does satisfy some of the vector space axioms, but not all of the axioms. Find two axioms that fail.

3. State the two step subspace test.

4. Let $W = \{(a, b, c) \in \mathbb{R}^3 : \text{ where } a + b + c = 1\}$.

   (a) Use the two step subspace test to show $(W, +, \cdot)$ is a subspace.
   (b) What geometric shape is W? Hint I gave it in standard form.
   (c) Give me the parametric for the geometric object defined in the set W.

5. Prove the following are subspaces of $V = \mathbb{R}^3$ or are not subspaces.

   (a) $\{(x, y, z)|x + y - z = 0\}$
   (b) $\{(x, y, z)|xyz = 1\}$
   (c) $\{(x, y, z)|x + y - z = 1\}$
   (d) $\{(x, y, z)|x + y - z = 0 \text{ and } 2x = z\}$

6. Let $W = \{(x, y, z) \in \mathbb{R}^3 : \text{ where } x - 3y - z = 0\}$.

(a) Use the two step subspace test to show $(W, +, \cdot)$ is a subspace.

(b) What geometric shape is W? Hint I gave it in standard form.

(c) Give me the parametric for the geometric object defined in the set W.

2 Linear Independence

7. Let $S = \{(1, 2, 1), (0, 1, 2), (0, -1, 0)\}$.

   (a) Is S linearly independent? (There is an easy test for this problem).

   (b) Is $(2, 2, 2) \in \mathrm{Span}(S)$? If yes what is a linear combination of the vectors in S that equals $(2, 2, 2)$?

   (c) Does S span $\mathbb{R}^3$?

8. Let $S = \{x, x + 2, x^3 - x - 1, x^3\}$ be a set in P_3.

   (a) Is S linearly independent?

   (b) Is $x^3 + x^2 + x + 1 \in \mathrm{Span}(S)$? If yes what is a linear combination of the polynomials in S that equals $x^3 + x^2 + x + 1$?

   (c) Is $4x^3 - 2x \in \mathrm{Span}(S)$? If yes what is a linear combination of the polynomials in S that equals $4x^3 - 2x$?

   (d) Does S span P_3?

9. Are the following lists linearly independent? Justify your answer.

   (a) $\mathrm{Span}((1, -2, 1), (1, 1, 1), (-2, 3, -2))$

   (b) $\{x, x - x^2, x + x^2\}$

   (c) $\{(1, -2, 1), (3, -12, 2), (1, 2, 3)\}$

3 Span, Basis

10. Let $B = \{(1, 2, 1), (0, 1, 2), (0, -1, 0)\}$.

    (a) Is B a basis for $\mathbb{R}^3$

    (b) Write the vector $(1, 0, -1)$ relative to the basis B.

    (c) Write the vector (a, b, c) relative to the basis B.

(d) Find the change of basis matrix from the standard basis to the basis B. (we called it $P_{\text{STANDARD}\to B}$ in class).

11. For the following system of linear equations.

$$\begin{array}{cccccc} 2x_1 & -2x_2 & +4x_3 & & -6x_5 & = 2 \\ & & x_3 & +6x_4 & & = 0 \end{array}$$

    (a) Find the solution set.
    (b) Find a basis for the solution set.
    (c) What is the dimension of that solution set?

12. For the following subspace of P_3

$$W = \{a + bx + cx^2 + dx^3 : a = -c \text{ and } b = c + d\}$$

    (a) Find a basis for W.
    (b) What is the dimension of that solution set?

13. Answer yes or no and justify your answer.
    (a) Is $(1, 2, 3) \in \text{Span}((1, -2, 1), (1, 1, 1))$?
    (b) Is $x^2 - 1 \in \text{Span}(x, x - x^2, x + x^2)$?
    (c) Is $(1, 2, 3) \in \text{Span}((1, -2, 1), (3, -12, 2))$?
    (d) Is $(1, 2, 3) \in \text{Span}((4, 5, 6), (7, 8, 9))$?

14. Find a a basis and dimension for the following.
    (a) $\text{Span}((1, -2, 1), (1, 1, 1), (-2, 3, -2))$
    (b) $\{p \in \mathcal{P}(\mathbb{R}) : p'(3) = 0\}$
    (c) $\{p \in \mathcal{P}_3(\mathbb{R}) : p'(3) = 0\}$
    (d) $\{(x, y, z, w) \in \mathbb{R}^4 : x + y - 2z = 0 \text{ and } 3y - z - w = 0\}$

4 Change of Basis Matrix

15. Let $B_1 = \{(-1, 1), (2, 3)\}$, $B_2 = \{(1, -1), (1, 1)\}$ and let B be the standard unit basis for $\mathbb{R}^2$.
    (a) Find the change of basis matrices for $P_{B_1 \to B_2}$ and $P_{B_1 \to B_2}$.
    (b) Find the coordinates of the point $(4, 6)$ (given in the standard basis) relative to the bases B_1 and B_2.

(c) Find the change of basis matrices for $P_{B \to B_2}$ and $P_{B_2 \to B}$.

(d) Find the coordinates of the point $(2, -4)$ (given in the standard basis) relative to the bases B and B_2. Graph this point the two separate coordinate axes B and B_2.

5 Row Space, Column Space & Null space

16. Let W be the plane $x - 2y + z = 0$ in $\mathbb{R}^3$.

    (a) Find the parametric equation for the plane.

    (b) Find a basis for W.

    (c) Compute the solution set to the linear system $x - 2y + z = 0$ in $\mathbb{R}^3$.

17. Let W be the hyperplane $x_1 - 2x_2 + x_3 + 6x_4 = 0$ in $\mathbb{R}^4$.

    (a) Find the parametric equation for the hyperplane.

    (b) Find a basis for W.

    (c) Compute the solution set to the linear system $x_1 - 2x_2 + x_3 + 6x_4 = 0$ in $\mathbb{R}^4$.

18. Let $A = \begin{bmatrix} -1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & -1 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{bmatrix}$.

    (a) Find a basis for the Column Space of A, COL(A).

    (b) Compute the dimension of COL(A).

    (c) Find a basis for the null space of A, NULL(A).

    (d) Compute the dimension of NULL(A).

19. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the formula $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x + y \\ y + z \\ x - z \end{bmatrix}$.

    (a) Find the matrix, A, to represent the linear transformation T.

    (b) Compute the basis for the Range of T, which is the Column Space of A.

(c) Find a basis for the null space of A, NULL(A).

(d) Compute the dimension of COL(A) and NULL(A). The dimension of the range of T is called the rank of T and the dimension of the null space is called the nullity.

(e) What is the dimension of the domain of T and the codomain of T? Again, compare Rank, Nullity and the dimension of the Domain. Do you see a relation?

6 Basic Transformations

20. Write the matrix for the following transformations described below.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is rotated by 45° counter-clockwise.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is reflected about the x-axis.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the x-axis is contracted by half and the y-axis is dilated by 2.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is rotated by 30° counter-clockwise and then reflected about the x-axis.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where the plane is reflected about the x-axis and then rotated by 30° counter-clockwise.

(f) $T : \mathbb{R}^3 \to \mathbb{R}^3$ where the x-axis is contracted by half and the z-axis is dilated by 2.