34. UNITARILY DIAGONALIZE THE MATRIX A. (Note A is Hermitian)

$$A = \begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix}$$

SOL'N:

Find eigen values

$$|A = I\lambda| = \begin{vmatrix} 4-\lambda & 1-i \\ 1+i & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda) - (1+1)$$

$$= 20 - 9\lambda + \lambda^2 - 2 = \lambda^2 - 9\lambda + 18$$

$$= (\lambda - 3)(\lambda - 6)$$

So $\lambda_1 = 3$ AND $\lambda_2 = 6$.

$\boxed{\lambda_1 = 3}$ $\quad A - 3\lambda = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \leftarrow$ FIND NULL SPACE $= \text{SPAN}(v_1)$

$$V_1 = \begin{bmatrix} -2 \\ 1+i \end{bmatrix}$$

$\boxed{\lambda_2 = 6}$ $\quad A - 6\lambda = \begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix}$

$$V_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

NOTE $\quad V_1 \cdot V_2 = (-2, 1+i) \cdot (1, 1+i) = -2\overline{(1)} + (1+i)\overline{(1+i)}$

$$= -2 + (1+1) = 0.$$

34. (CONTINUED)

$u_1 = \frac{1}{\|v_1\|} v_1$ AND $u_2 = \frac{1}{\|v_2\|} v_2$

$= \frac{1}{\sqrt{2^2 + |1+i|^2}} (-2, 1+i)$

$= \frac{1}{\sqrt{4 + (\sqrt{2})^2}} (-2, 1+i)$

$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$= \left(\frac{-2}{\sqrt{6}}, \frac{1+i}{\sqrt{6}} \right)$

$u_2 = \frac{1}{\sqrt{1^2 + |1+i|^2}} (1, 1+i)$

$= \frac{1}{\sqrt{3}} (1, 1+i)$

$= \left(\frac{1}{\sqrt{3}}, \frac{1+i}{\sqrt{3}} \right)$

SO $P = \begin{bmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{1+i}{\sqrt{3}} \end{bmatrix}$ AND $D = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$.

(32.) Make A Hermitian & verify it's eigenvalues are real.

$$A = \begin{bmatrix} 3 & 2-3i \\ ? & -1 \end{bmatrix}$$

To be Hermitian we need

$A^* = A$. Note $A^* = \begin{bmatrix} 3 & \overline{?} \\ 2+3i & -1 \end{bmatrix}$

AND $A = \begin{bmatrix} 3 & 2-3i \\ ? & -1 \end{bmatrix}$

So $A = \begin{bmatrix} 3 & 2-3i \\ 2+3i & -1 \end{bmatrix}$

Now we find the eigen values of A.

$$|A-\lambda I| = \det\begin{bmatrix} 3-\lambda & 2-3i \\ 2+3i & -1-\lambda \end{bmatrix} = (3-\lambda)(-1-\lambda) - (2-3i)(2+3i)$$

$$= \left[-3-3\lambda+\lambda+\lambda^2\right] - (4+9)$$

$$= \lambda^2 - 2\lambda - 16$$

FIND THE ROOTS

$$\lambda = \frac{2 \pm \sqrt{4-(1)(-16)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{20}}{2}$$

and these are clearly in $\mathbb{R}$

(33.) Show A is unitary and find A^{-1}.

Recall A is unitary $\Leftrightarrow$ $A \cdot A^* = A^* A = I$.

$$A \cdot A^* = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} i \\ -\frac{4}{5} & \frac{3}{5} i \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} i & -\frac{3}{5} i \end{bmatrix}$$

Recall $A^* = \bar{A}^T$

$$= \begin{bmatrix} \frac{9}{25} - \frac{16}{25} i^2 & \frac{-12}{25} + \frac{-12}{25} i^2 \\ \frac{-12}{25} - \frac{12}{25} i^2 & \frac{16}{25} - \frac{9}{25} i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So A is unitary & $A^* = A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} i & -\frac{3}{5} i \end{bmatrix}$

36. Let $B = \frac{1}{2}(A + A^*)$ AND $C = \frac{1}{2i}(A - A^*)$

(a) Show B is Hermitian.

$\overline{\frac{1}{2}} = \frac{1}{2}$ $A^{**} = A$.

$$B^* = \left[\frac{1}{2}(A + A^*)\right]^*$$

$$= \overline{\frac{1}{2}}\left(A^* + A^{**}\right) = \frac{1}{2}(A^* + A) = B$$

So $B^* = B$. Thus B is Hermitian.

(b) Show $A = B + iC$ AND $A^* = B - iC$

$$B + iC = \frac{1}{2}(A + A^*) + i\left(\frac{1}{2i}\right)(A - A^*)$$

$$= \frac{1}{2}A + \frac{1}{2}A^* + \frac{1}{2}A - \frac{1}{2}A^* = A.$$

$B - iC$ is for you.

(c) What condition must B & C satisfy for A to be normal?

from part (b)

Soln Note $A \cdot A^* = (B + iC)(B - iC) = B^2 - iBC + iCB + C^2$

AND $A^* \cdot A = (B - iC)(B + iC) = B^2 + iBC - iCB + C^2$

36. (continued) So if $A^*A = AA^*$ then

$$\cancel{B^2} + -iBC + iCB + \cancel{C^2} = \cancel{B^2} + iBC - iCB + \cancel{C^2}$$

$$\cancel{2i}\,CB = \cancel{2i}\,BC$$

$$\boxed{CB = BC}$$

37. just computation

38. $A = A^{-1}$.

$A = A^*$ ← Hermitian

$AA^* = I$ so $A = A^{-1}$