(24) (continued)

$$u_1 = \frac{1}{\|w_1\|} \cdot w_1 = \frac{1}{\sqrt{5}}(-2, 1, 0) = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$u_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\cancel{\frac{1}{5}}(18^2+4^2+1)^{1/2}} \cancel{\tfrac{1}{5}}(18, 4, 1) = \frac{1}{\sqrt{341}}(18, 4, 1) = \left(\frac{18}{\sqrt{341}}, \frac{4}{\sqrt{341}}, \frac{1}{\sqrt{341}}\right)$$

So $D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} -1/3 & -2/\sqrt{5} & 18/\sqrt{341} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{341} \\ 2/3 & 0 & 1/\sqrt{341} \end{bmatrix}$

AND $\boxed{Q = 10y_1^2 + y_2^2 + y_3^2}$.

1. (a) $W = \{(x,y) \mid x+y=0\}$

So $W = \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ ⟵ from the solution set S of W

Thus $W^\perp = \{(x,y)\in\mathbb{R}^2 \mid (x,y)\cdot(1,-1)=0\}$

$(x,y)\cdot(1,-1)=0$

$x + -y = 0$

↓

$\underline{W^\perp = S = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}}$.

(b) $W = \{(x,y,z) \mid x+y=0\}$

$x+y=0$

$y=s \quad z=t$

So $W = S = \left\{\begin{bmatrix}-s\\s\\t\end{bmatrix} ; s,t\in\mathbb{R}\right\}$

$= \text{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Thus $W^\perp = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y,z)\cdot(-1,1,0)=0$ AND $(x,y,z)\cdot(0,0,1)=0\}$

$\left.\begin{aligned}-x+y &= 0\\ z &= 0\end{aligned}\right] \xrightarrow{\text{solution}} W^\perp = S = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$

(c) $W = \{(x,y,z) \mid x+y=0$ AND $3y-z=0\}$

$\left.\begin{aligned}x+y&=0\\3y-z&=0\end{aligned}\right] \to$ FIND S

So $W = S = \text{span}\left\{\begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$

Thus $W^\perp = \{(x,y,z)\in\mathbb{R}^3 \mid (-1,1,3)\cdot(x,y,z)=0\}$

$(-1,1,3)\cdot(x,y,z)=0$

$-x+y+3z=0$ ⟶ FIND $S = W^\perp = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1/3\\0\\1\end{bmatrix}\right\}$

(2.) For You

(3.) Prove $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$

Proof.
$$\langle u+v, u-v\rangle = \langle u+v, u\rangle + \langle u+v, -v\rangle$$
$$= \langle u, u\rangle + \langle v, u\rangle + \langle u, -v\rangle + \langle v, -v\rangle$$
$$= \|u\|^2 + \langle u, v\rangle - \langle u, v\rangle + - \langle v, v\rangle$$
$$= \|u\|^2 \qquad - \|v\|^2. \quad \boxed{R}$$

(4.) Prove $\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$.

Proof.
Note
$$\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = \frac{1}{4}\langle u+v, u+v\rangle - \frac{1}{4}\langle u-v, u-v\rangle$$
$$\vdots$$
$$= \langle u, v\rangle. \quad \boxed{R}$$

(5) in class

(6.) Show u is orthogonal to w, where
$$w = v - \frac{\langle u, v\rangle}{\langle u, u\rangle} u$$

Proof
$$\langle u, w\rangle = \langle u, v - \frac{\langle u, v\rangle}{\langle u, u\rangle} u\rangle$$
$$= \langle u, v\rangle - \langle u, \frac{u \; v}{u \; u} u$$
$$= \langle u, v\rangle - \frac{\langle u, v\rangle}{\langle u, u\rangle}\langle u, u\rangle \qquad \text{note } \frac{\langle u, v\rangle}{\langle u, u\rangle} \text{ is a scalar}$$
$$= 0$$

Thus u is orthogonal to w.

7. Assume $\|Tv\| \leq \|v\|$ $\forall v \in V$. Show

$T - \sqrt{2} I$ is invertible

Hint: λ is an eigenvalue of $T \Leftrightarrow T - \lambda I$ is NOT invertible

Proof Let λ be an eigenvalue of T. So

$Tv = \lambda v$ where v is a corresponding eigenvector

Thus $\|Tv\| = \|\lambda v\| = |\lambda| \, \|v\|$ So $|\lambda| = \frac{\|Tv\|}{\|v\|}$. Since $\|Tv\| \leq \|v\|$

we have $|\lambda| \leq 1$. So all eigenvalues of T

satisfy $|\lambda| \leq 1$. Thus $\sqrt{2}$ is NOT an eigenvalue of T

So $T - \sqrt{2} I$ is invertible. ∎

(8.), (9.) Like in class

(10) Find $U^{\perp}$ for the following sets below.

(a) $U = \{(1,0,1)\} \leq \mathbb{R}^3$ usual Euclidean inner product

$U^{\perp} = \{(x\ y\ z) \in \mathbb{R}^3 \mid (x, y, z) \cdot (1, 0, 1) = 0\}$

Thus $(x, y, z) \cdot (1, 0, 1) = 0$

$x + z = 0$ so $U^{\perp} = S = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(12.) Let $T(u) = 3w$ AND $T(w) = 3u$

where u, w are nonzero vectors

Show either 3 or -3 is an eigenvalue of T.

Proof. Either $u = -w$ OR $u \neq -w$.

CASE I Assume $u = -w$, Then define $V = u - w$. Note $V \neq 0$.

Then $T(V) = T(u-w) = Tu - Tw$

$= 3w - 3u$

$= 3(w-u)$

$= -3(u-w) = -3V$

So $TV = -3V$ THUS -3 is an eigenvalue for T

CASE II Assume $u \neq -w$. Then define $V = u + w$. Note $V \neq 0$.

Then $T(V) = T(u+w) = Tu + Tw = +3w + 3u$

$= 3(w+u) = 3(u+w) = 3V$.

So $TV = 3V$ thus 3 is an eigenvalue for T. ∎

MY SOL

$Tu = 3w$

$T(Tu) = T(3w) = 3Tw = 3u$ → So

$(T-3I)u = 0$ OR

$(T-3I)u \neq 0$

⋮

$T^2 u = 3u$

$T^2 u - 3u = 0$

$(T^2 - 3I)u = 0$

$(T+3I)(T-3I)u = 0$

(10) (b) $U = \{1\} \subseteq \mathbb{P}_2$. with $\langle f, g\rangle = \int_0^1 f g \, dx$

$U^\perp = \{ a_0 + a_1 x + a_2 x^2 \mid \langle 1, a_0 + a_1 x + a_2 x^2 \rangle = 0 \}$

So $\langle 1, a_0 + a_1 x + a_2 x^2 \rangle = \int_0^1 1 (a_0 + a_1 x + a_2 x^2) \, dx$

$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \Big|_0^1 = a_0 + \frac{1}{2} a_1 + \frac{1}{3} a_2 = 0$

Solve system $a_0 + \frac{1}{2} a_1 + \frac{1}{3} a_2 = 0$

$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 0 \end{array}\right]$ $U^\perp = S = \text{span} \left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(11) Find the Eigen values for

(a) $T \in \mathcal{L}(P_2(\mathbb{R}))$ where $T(p) = p'$

$T(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$

So $T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} \longrightarrow$ So $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$|T - \lambda I| = \left| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = \left| \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} \right| = \lambda^3$

eigen value
$\lambda = 0$
eigen vector
$p_1 = x^2$

So $\underline{\lambda = 0}$ $T - \lambda I = T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Null}(T - \lambda I) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span} \{ x^2 \}$

(b) $T \in \mathcal{L}(P_2(\mathbb{R}))$ where $Tp = xp'$

$T(a_0 + a_1 x + a_2 x^2) = x \cdot (0 + a_1 + 2a_2 x) = a_1 x + 2a_2 x^2$

$T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 2a_2 \end{bmatrix}$ So $T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(11)(b) (CONTINUED)

$$|T-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -\lambda(1-\lambda)(2-\lambda)$$

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$

$\boxed{\lambda_1 = 0}$ NULL $(T - 0I) = \underset{\text{WORK MISSING}}{\ldots} = \text{SPAN}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = \text{SPAN}\{1\}$

as vector — as a polynomial

$\boxed{\lambda_2 = 1}$ NULL $(T-I) = $ NULL $\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \underset{\text{WORK MISSING}}{\ldots} = \text{SPAN}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\} = \text{SPAN}\{x\}$

$\boxed{\lambda_3 = 2}$

NULL $(T-2I) \overset{\text{NULL}}{=} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{NULL}(T-2I) = \text{SPAN}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \text{SPAN}\{x^2\}.$$

eigen value	vector
$\lambda_1 = 0$	$V_1 = 1$
$\lambda_2 = 1$	$V_2 = x$
$\lambda_3 = 2$	$V_3 = x^2$

(13) Find LS solution and error vector. Verify error vector is orthogonal to solution space.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$Ax = b$

$A^T A x = A^T b$

$x = (A^T A)^{-1} A^T b = \left(\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 1.8181\ldots \\ -.7272\ldots \end{bmatrix}$

$LS = Ax - b = \begin{bmatrix} 0.545\ldots \\ 2.454\ldots \\ -1.363\ldots \end{bmatrix}$

$\|Ax - b\| = 2.86\ldots$

$A^T(Ax - b) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ← So ORTHOGONAL

I used R and my R code was

```
A= matrix(c(1,2,4,-1,3,5), nrow=3, ncol=2)
b =matrix(c(2,-1,5), nrow=3, ncol=1)
C = solve( t(A) %*% A)
```

Find inverse, transpose, matrix mult

```
x= C %*% t(A) %*% b
LS= A %*% x -b
t(A)%*% LS
norm(LS, type='2')
```

(14) Find the LS straight line $y = ax + b$ for

(a) $(0,0)$, $(1,2)$, $(2,7)$

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

So $x = (A^T A)^{-1} \cdot A^T b$ where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$

$$= \begin{bmatrix} -0.5 \\ 3.5 \end{bmatrix} \begin{matrix} \leftarrow b \\ \leftarrow a \end{matrix}$$

So $\boxed{y = 3.5x - 0.5}$ is the LS line with

error $= 1.22\ldots$

(15.) LS quadratic fit for $y = a_0 + a_1 x + a_2 x^2$ for

(a) $(2,0)$, $(3,10)$, $(5,48)$ $(6,76)$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}}_{A} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 10 \\ 48 \\ 76 \end{bmatrix}}_{b}$$

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$$

So $\boxed{y = -2 + -5x + 3x^2}$ is the LS quadratic

with error near zero

(16.) $A = \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$

(1) $A^T A = \begin{bmatrix} 4/5 & -9/25 & 12/25 \\ 0 & 4/5 & 3/5 \\ -3/5 & -12/25 & 16/25 \end{bmatrix} \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix}$

$= \cdots$ some work $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

(a) Row vectors

$r_1 = \left(\frac{4}{5}, 0, -\frac{3}{5}\right)$ $\quad r_2 = \left(-9/25, 4/5, -12/25\right)$

$r_3 = \left(12/25, 3/5, 16/25\right)$

Note $r_1 \cdot r_1 = \frac{16}{25} + 0 + \frac{9}{25} = \frac{25}{25} = 1$

$r_1 \cdot r_2 = \left(\frac{4}{5}\right)\left(\frac{-9}{25}\right) + (0)\left(\frac{4}{5}\right) + \left(\frac{-3}{5}\right)\left(-\frac{12}{25}\right)$

$= \frac{-36}{125} + 0 + \frac{36}{125} = 0$

$r_1 \cdot r_3 = \left(\frac{4}{5}\right)\left(\frac{12}{25}\right) + (0)\left(\frac{3}{5}\right) + \left(\frac{-3}{5}\right)\left(\frac{16}{25}\right) = \frac{48}{125} - \frac{48}{125}$

$= 0.$

$r_2 \cdot r_2 = \qquad = 1$

$r_2 \cdot r_3 = \cdots \quad = 0$

$r_3 \cdot r_3 = \cdots \quad = 1$

work missing

(b) similar to (a)

(17) $T_A(x) = \begin{bmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} \frac{8}{5} + 0 - \frac{15}{5} \\ \frac{-18}{25} + \frac{12}{5} - \frac{60}{25} \\ \frac{24}{25} + \frac{9}{5} + \frac{80}{25} \end{bmatrix} = \begin{bmatrix} 7/5 \\ -18/25 \\ 149/25 \end{bmatrix}$$

24 + 45 + 80

125
24

Note $\|x\| = \sqrt{2^2 + 3^2 + 5^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$

AND $\|T_A x\| = \|\left(\frac{7}{5}, \frac{-18}{25}, \frac{149}{25}\right)\| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{-18}{25}\right)^2 + \left(\frac{149}{25}\right)^2}$

$$= \sqrt{\frac{49 \cdot 25 + 18^2 + 149^2}{625}} = \sqrt{38}.$$

(18) What conditions must $a, b \in \mathbb{R}$ satisfy for $A = \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}$ to be orthogonal?

$c_1 = (a+b, a-b)$ $\quad c_2 = (a-b, a+b)$

we need
$c_1 \cdot c_1 = 1$ $\quad c_1 \cdot c_2 = 0$ AND $c_2 \cdot c_2 = 1$

$c_1 \cdot c_1 = (a+b)^2 + (a-b)^2$
$= a^2 + 2ab + b^2 + a^2 - 2ab + b^2$
$= 2a^2 + 2b^2 = 1$

(1) $\boxed{a^2 + b^2 = \frac{1}{2}}$

$c_1 \cdot c_2 = (a+b)(a-b) + (a-b)(a+b)$
$= a^2 - b^2 + a^2 - b^2$
$= a^2 - b^2 = 0$

(2) $a = \pm b$.

(19.) UNDER WHAT CONDITIONS WILL A DIAGONAL MATRIX BE ORTHOGONAL?

SOL'N all diagonal entries are ± 1.

(20) Show Note x is an $n \times 1$ matrix

$$A = I_n - \frac{2}{x^T x} x x^T$$

is orthogonal AND symmetric.

To show A is orthogonal compute $A^T A$.

$$A^T A = \left[I - \frac{2}{x^T x} x x^T\right]^T \left[I - \frac{2}{x^T x} x x^T\right]$$

$$= \left[I^T - \frac{2}{x^T x}(x x^T)^T\right]\left[I - \frac{2}{x^T x} x x^T\right]$$

note $\frac{2}{x^T x}$ is a scalar

$$= \left[I - \frac{2}{x^T x}\left(x^{T^T} \cdot x^T\right)\right]\left[I - \frac{2}{x^T x} x x^T\right]$$

$$= \left[I - \frac{2}{x^T x}(x x^T)\right]\left[I - \frac{2}{x^T x} x x^T\right]$$

$$= \left[I^2 - I\frac{2}{x^T x} x x^T + -\frac{2}{x^T x} x x^T I + \left(\frac{2}{x^T x}\right)^2 x x^T \cdot x x^T\right]$$

$$= I - \frac{4}{x^T x} x x^T + \frac{4}{x^T x \cdot x^T x} x (x^T x) x^T$$

$$= I - \frac{4}{x^T x} x x^T + \frac{4}{x^T x} \cdot x x^T = I.$$

To show A is symmetric Compute A^T AND note $A^T = A$.

(21) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal

Let v, w be not orthogonal, and non-zero.

Thus $0 \neq v \cdot w = Av \cdot Aw$ by hint

since v is NOT orthogonal to w

So $Av \cdot Aw \neq 0$

THUS Av is NOT orth. to Aw.

(22) FIND THE SPECTRAL DECOMPOSITION OF THE FOLLOWING

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(1) FIND Eigen Values

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$$3 - \lambda = \pm 1$$

$$\lambda = 3 \mp 1 = 4 \text{ or } 2$$

$\lambda_1 = 4 \quad \lambda_2 = 2$

(2) FIND EIGEN VECTORS

$\boxed{\lambda_1 = 4}$ $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ so $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\boxed{\lambda_2 = 2}$ $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

THUS

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T =$$

$$A = 4 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(23.) For you read and understand.

(24.) FIND ORTH CHANGE OF BASIS MATRIX TO ELIMINATE THE CROSS TERMS. EXPRESS Q in terms of new variables

(a) $Q = 2x_1^2 + 2x_2^2 - 2x_1x_2$

SOL'N: $Q = x^T \cdot A x$ where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(1) EIGEN VALUES OF A are $\lambda_1 = 1$ AND $\lambda_2 = 3$

(2) EIGEN VECTORS $\lambda_1 = 1$ $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\lambda_2 = 3$ $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

(3) $D = P^T A P$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ AND $P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$Q = y^T D y = y_1^2 + 3y_2^2$ ← an ellipse

(b) eigen values 6, 4, 1

(c) eigen values 7, 4, 1

(d) $Q = 2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3$

$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$ → I get $\lambda_1 = 10$, $\lambda_2 = 1$ ← repeated

24 (continued)

$\boxed{\lambda_1 = 10}$ $A - 10I = \begin{bmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -8 & 2 & -2 & 0 \\ 2 & -5 & -4 & 0 \\ -2 & -4 & -5 & 0 \end{array}\right] \xrightarrow{R.R.} \text{some work} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free

$z = t$

$$S = \left\{ \begin{bmatrix} -\frac{1}{2}t \\ -t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad V_1 = \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} \qquad u_1 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$u_1 = \frac{1}{\|V_1\|} \cdot V_1 = \frac{1}{\sqrt{\frac{1}{4}+1+1}} \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{\frac{9}{4}}} \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\lambda_2 = 1}$ $A - 1I = \begin{bmatrix} 1 & 2 & -2 \\ 2 & +4 & -4 \\ -2 & -4 & +4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 4 & -4 & 0 \\ -2 & -4 & 4 & 0 \end{array}\right] \xrightarrow{RR} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad S = \left\{ \begin{bmatrix} -2s+2t \\ s \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t : s,t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$V_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ | $w_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$V_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ | $w_2 = V_2 - \dfrac{\langle V_2, w \rangle}{\langle w_1, w_1 \rangle} w_1 = (2,0,1) - \left(\dfrac{(2,0,1)\cdot(-2,1,0)}{(-2,1,0)\cdot(-2,1,0)}\right)(-2,1,0)$

$$= (2,0,1) - \left(\frac{-4+0+0}{4+1+0}\right)(-2,1,0) = (2,0,1) + \left(\frac{8}{5}, \frac{4}{5}, 0\right)$$

$$= \left(\frac{18}{5}, \frac{4}{5}, 1\right) = \frac{1}{5}(18,4,1)$$

(26) Determine if matrix is pos. def, neg. def or indefinite.

We have two ways to test for definiteness for sym. matrix

(a) FIND EIGEN VALUES

(b) FIND Determinants of principal submatrices

(a)

$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{bmatrix}$ $\quad |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & -1-\lambda & 3 \\ 2 & 3 & 2-\lambda \end{vmatrix}$

$= (3-\lambda)\left((-1-\lambda)(2-\lambda) - 9\right) - 1\left(2-\lambda - 6\right) + 2\left(3 - 2(-1-\lambda)\right)$

$= (3-\lambda)\left[-2-\lambda+\lambda^2-9\right] - \left[-4-\lambda\right] + 2\left[3+2+2\lambda\right]$

$= (3-\lambda)\left[\lambda^2-\lambda-11\right] + (4+\lambda+10+4\lambda)$

$= 3\lambda^2 - 3\lambda - 33 + -\lambda^3 + \lambda^2 + 11\lambda + (14 + 5\lambda)$

$= -\lambda^3 + 4\lambda^2 + 13\lambda - 33$

with technology I get $\lambda_1 = 5.69\ldots$
$\lambda_2 = 1.16\ldots$
$\lambda_3 = -2.86\ldots$

So A is indefinite.

(b) $|[3]| = 3$

$\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -3 - 1 = -4$ $\quad \longrightarrow$ indefinite

$\begin{vmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{vmatrix} = -1$