Math 3520 - Test 1 Review

1 Preliminaries and Induction

1. Induction, definition of odd, even, sets union, intersection, complement

2. Prove if $3|n^2$ then $3|n$. Use the contrapositive and use division algorithm (that is cases for each possible remainder).

3. Prove If n is odd then $4|n^2 - 1$ for all $n \in \mathbb{Z}$ using division algorithm (that is cases for each possible remainder).

4. Prove with induction that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

5. Prove $8|5^{2n} - 1$ for all $n \in \mathbb{N}$ using induction.

6. Prove $n! > 2^n$ for all $n \in \mathbb{N}$ with $n > 3$.

7. Define the sequence (induction)

$$a_1 = 1 \text{ and } a_n = \sqrt{a_{n-1} + 3}$$

   (a) Prove a_n is increasing. That is, prove $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

   (b) Prove a_n is bounded above by 6. That is, prove $a_n \leq 6$ for all $n \in \mathbb{N}$.

8. Which of the following are well ordered?

   - $A = \{1, 2, 3\}$
   - $B = \{-n | n \in \mathbb{N}\}$
   - $C = \{p | p \in \mathbb{Z} \text{ and p is prime}\}$
   - $\mathbb{Z}$
   - $E = \{n | n \in \mathbb{N} \text{ and } 3|n\}$
   - $F = \{n | n \in \mathbb{Z} \text{ and } 3|n\}$
   - $G = \{n | n \in \mathbb{N} \text{ and } 3 \leq n \leq 45\}$
   - $H = \{n | n \in \mathbb{Q} \text{ and } 3 \leq n \leq 45\}$
   - $I = \{1 - \frac{1}{n} | n \in \mathbb{N}\}$

9. Prove: If A is well ordered and $B \subseteq A$ then B is well ordered.

2 Relations

10. Define relation, equivalence relation, well ordered, reflexive, symmetic, transitive, domain, codomain, partition of a set

11. We define the given relation from A to B by

$$R = \{(1, a), (2, a), (3, a), (4, a), (1, a), (2, a)\}$$

where $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. What are the domain and codomain?

12. We define the given relation on A by

$$R = \{(1, 1), (2, 1), (3, 1), (4, 1), (1, 2), (2, 2)\}$$

where $A = \{1, 2, 3, 4\}$.

   (a) What are the domain and codomain?
   (b) Is R reflexive? If it is not reflexive, expand R so that it is reflexive.
   (c) Is R symmetric? If it is not symmetric, expand R so that it is symmetric.
   (d) Is R transitive? If it is not transitive, expand R so that it is transitive.
   (e) Is R an equivalense relation?

13. We define the given relation on A by

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 2), (2, 3), (a, b)(c, d)\}$$

where $A = \{1, 2, 3, 4\}$.

   (a) Assume R is an equivalence relation. Find (a, b) and (c, d).
   (b) What are the equivalence classes for R.

14. We define the given relation on $\mathbb{Z}$ by

$$aRb \Leftrightarrow 4|3a - b.$$

   (a) Prove or disprove R is reflexive.
   (b) Prove or disprove R is symmetric.
   (c) Prove or disprove R is transitive.

(d) What are the equivalence classes for R?

15. Let $a, b \in \mathbb{Z}$ and let

$$aRb \Leftrightarrow 3|a - b$$

be a relation on $\mathbb{Z}$. Prove R is an equivalence relation on $\mathbb{Z}$. And determine the distinct equivalence classes.

16. We define the given relation on A by

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,2),(2,3),(a,b)(c,d)\}$$

where $A = \{1,2,3,4\}$.

(a) Assume R is an equivalence relation. Find (a, b) and (c, d).

(b) What are the equivalence classes for R.

17. We define the given two relations below on $\mathbb{Z}$. Answer the questions for each of them. One relation is an equivalence relation the other is not an equivalence relation

- $aRb \Leftrightarrow 4|3a - b$.
- $aRb \Leftrightarrow 4|3a + b$.

(a) Prove R is reflexive.

(b) Prove R is symmetric.

(c) Prove R is transitive.

(d) What are the equivalence classes for R.

18. Let R_1 and R_2 be equivalence relations on the set A. Prove that $R = R_1 \cap R_2$ is also an equivalence relations on the set A.

19. Construct the addition and multiplication tables for $\mathbb{Z}_4$ and $\mathbb{Z}_5$.

3 Functions

20. Define function, injective, surjective, domain, codomain, range, inverse image, inverse function, permutations.

21. Define $f : \{1,2,3\} \rightarrow \{4,7,9\}$ by $f(1) = 4$, $f(2) = 4$ and $f(3) = 9$.

(a) Is f injective, surjective or bijective?

(b) Compute $f(\{1,2\})$, $f^{-1}(\{1,4\})$ and $f \circ f^{-1}(\{4,9\})$.

22. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(n) = 2n - 1$. Is f injective, surjective or bijective? Prove or disprove.

23. Define $f : (-\infty, 0) \to [0, \infty)$ by $f(x) = x^2$.

    (a) Is f injective, surjective or bijective? Prove or disprove.

    (b) Compute $f((-2,2))$, $f^{-1}((-2,2))$ and $f^{-1} \circ f(\{4,9\})$.

    (c) Compute $f^{-1} \circ f(\{4,9\})$ and $f \circ f^{-1}(\{4,9\})$.

24. Define $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}$ by $f(x) = \frac{x}{x-2}$. Is f injective, surjective or bijective? Prove or disprove.

25. Let $f : A \to B$ and $g : B \to C$. Prove the following.

    (a) If f and g are injective then $g \circ f$ is injective.

    (b) If f and g are surjective then $g \circ f$ is surjective.

26. Find all bijective functions from $\{1,2,3\}$ to $\{1,2,3\}$. How many functions did you come up with?

27. List all elements of the set $\mathcal{S}_3$. How many elements are in the set $\mathcal{S}_6$?

28. For the following permutaions:
    $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$
    compute:

    (a) $\sigma_1 \circ \sigma_1$

    (b) $\sigma_1 \circ \sigma_2 \circ \sigma_2$

    (c) σ_1^3

    (d) σ_1^{-1}

4 Cardinality

29. Define cardinality. That is $A \sim B$ if and only if ...

30. Know $\sim$ is an equivalence relation and what that means.

31. Show $\mathbb{N} \sim 2\mathbb{N}$

32. Let $A = [0, 2]$ and $B = [-1, 6]$. Show $f : A \to B$ given by $f(x) = \frac{7}{2}x - 1$ is a bijection. What does this tell uas about the sets A and B?

33. Show the sets have the same cardinality

    (a) $\mathbb{N}$ and $\mathbb{Z}$

    (b) ~~$\mathbb{N}$ and $\mathbb{Q}$~~

    (c) $\{a, b, c\}$ and $\{1, 2, 7\}$

    (d) $\mathbb{N} \times \{1, 2, 3, 4\}$ and $\mathbb{N}$

    (e) $[0, 1)$ and $(2, 3]$

    (f) $(0, 1)$ and $\mathbb{R}$

34. ~~Show the sets **do not** have the same cardinality~~

    (a) ~~$\mathbb{N}$ and $\mathbb{R}$.~~

    (b) ~~$\mathbb{N}$ and the set of irrationals.~~