1. The definition

$$\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{so that}$$

$$\text{if} \quad x \in \text{Domain}, \ 0 < |x - c| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon.$$

2. (b)

$$\lim_{x \to -3} x^2 + 1 = 10$$

SW

$\boxed{\delta \le 1}$

$|x - (-3)| < \delta \le 1$

$-1 < x + 3 < +1$

$-3 \quad\quad -3 \quad\quad -3$

$-4 < x < -2$

So $|x| < 4$

& $\boxed{\delta \le \varepsilon/7}$

$|x - c| < \delta$

$$|f(x) - L| = |x^2 + 1 - 10|$$

$$= |x - 3|\,|x + 3|$$

$$\le (|x| + 3)\,|x + 3| \quad \text{by } T_1$$

$$< (4 + 3)\,|x + 3|$$

$$< 7 \cdot \frac{\varepsilon}{7}$$

$\boxed{\delta \le \varepsilon/7}$

So $\delta \le 1$ AND $\delta \le \varepsilon/7$

Proof. Let $\varepsilon > 0$. Let $\delta = \min\{1, \varepsilon/7\}$. Let $x \in \mathbb{R}$ so that $0 < |x - (-3)| < \delta$. Thus $|x + 3| < 1$

so $-1 < x + 3 < 1$

$-4 < x < -2$. So $|x| < 4$ $(*)$.

CONTINUED

(2b) (CONTINUED)

Note $|f(x) - L| = |x^2+1 - 10| = |x^2-9|$

$= |x-3|\ |x+3|$

$\leq (|x| + 3)\ |x+3|$ by TRIANGLE INEQUALITY

$< (4+3)\ |x+3|$ by $(*)$.

$< 7 \cdot \delta$ since $|x+3| < \delta$

$\leq 7\ (\varepsilon/7) = \varepsilon$. ☒

(2d) $\lim_{x \to 2} x^3 = 8$.

$\boxed{\delta \leq 1}$

$|x-c| < \delta$

$|x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

So $|x| < 3\ (*)$

$|x-c| < \delta$

$|f(x) - L| = |x^3 - 8| = |x-2|\,|x^2+2x+4|$

$\leq |x-2|\,(|x|^2 + 2|x| + 4)$ by TI

$< |x-2|\,(3^2 + 2(3) + 4)$ by $*$

$< \delta \cdot 19$

$\leq \boxed{\frac{\varepsilon}{19}} \cdot 19$

$\delta \leq 1$ $\quad$ $\delta \leq \varepsilon/19 \Rightarrow \delta = \min\{1, \varepsilon/19\}$

__Proof.__ Let $\varepsilon > 0$. Define $\delta = \min\{1, \varepsilon/19\}$. Let $x \in \mathbb{R}$ so that $0 < |x-2| < \delta$. Thus $|x-2| < 1$

$-1 < x-2 < +1$

$1 < x < 3$. So $|x| < 3\ (*)$.

So $|f(x) - L| = |x^3 - 8| = |x-2|\,|x^2+2x+4| \leq |x-2|\,(|x|^2 + 2|x| + 4)$ by T.I

$< |x-2|\,(9+6+4)$ by $(*)$

$< \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. ☒

7. Let G be a group, and let g be a fixed element in G.

Define $H = \{ g a g^{-1} \mid a \in G \}$

Show H is a subgroup of G.

Proof. We will use the 2-step subgroup test to show H is a subgroup of G. First note

$$g e g^{-1} = e \in H \quad \text{for } a = e.$$

So H is nonempty.

Let $x, y \in H$. So $x = g a g^{-1}$ and $y = g b g^{-1}$ where $a, b \in G$. Note

$$x \cdot y = g a g^{-1} g b g^{-1} = g a b g^{-1}$$

Since $ab \in G$ we have $xy = g(ab)g^{-1} \in H$.

Now we will show $x^{-1} \in H$. Since $x = g a g^{-1}$ we have

$$x^{-1} = (g a g^{-1})$$
$$= (g^{-1})^{-1} a^{-1} g^{-1}$$
$$= g a^{-1} g^{-1}, \text{ Note } a^{-1} \in G.$$

So $x^{-1} \in H$.

By 2-step subgroup test H is a subgroup of G. ∎

(9.) Let $G_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ AND $G_2 = (\mathbb{Z}_8, +)$

(a) Find orders of $(1,1,1)$ AND $(1,0,1)$ in G_1

$(1,1,1) + (1,1,1) = (0,0,0)$ ← the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

So $|(1,1,1)| = 2$

Similarly $|(1,0,1)| = 2$

The orders of $1, 2, 3$ in $\mathbb{Z}_8$

$1 = 1$ $1+1 = 2$ $1+1+1 = 3$ $\vdots$ $1 + \cdots + 1 = 8 \equiv 0$ $|1| = 8$	2 $2+2$ $2+2+2$ $2+2+2+2 = 8 \equiv 0$ $|2| = 4$	AND $|3| = 8$

(b) G_1 has NO generators

but $1, 3, 5, 7$ are all generators for $(\mathbb{Z}_8, +)$.

(c) $G_1 \not\cong G_2$ since G_1 has NO generator

but G_2 does have a generator.

(10) $G_1 = (\mathbb{Z}_9^*, \cdot)$ $\quad$ $G_2 = (\mathbb{Z}_6, +)$

(a) $2, 5, 7 \in \mathbb{Z}_9^*$

$2 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16 \equiv 7$
$2^5 = 14 \equiv 5$
$2^6 \equiv 10 \equiv 1$

$|2| = 6$

$5 = 5$
$5^2 \equiv 25 \equiv 7$
$5^3 \equiv 35 \equiv 8$
.
$5^6 \equiv 1 \mod 9$

$|5| = 6$

7
$7^2 \equiv 49 \equiv 4$
$7^3 \equiv 4 \cdot 7 \equiv 28$
$\equiv 1 \mod 9$

$|7| = 3$

$1, 2, 3 \in \mathbb{Z}_6$

$|1| = 6$, $|2| = 3$, $|3| = 2$

(b) 2 is a generator in $\mathbb{Z}_9^*$

AND 1 is a generator in $\mathbb{Z}_6$

(c)
$2 = 2 \longrightarrow 1 = 1$
$2 = 4 \longrightarrow 1+1 = 2$
$2^3 \equiv 8 \longrightarrow 1+1+1 = 3$
$2^4 \equiv 7 \longrightarrow 1+1+1+1 = 4$
$2^5 \equiv 5 \longrightarrow 1+1+1+1+1 \equiv 5$
$2^6 \equiv 1 \longrightarrow 1+ \cdots +1 = 6 \equiv 0$

So

$\mathbb{Z}_9^* \xrightarrow{f} \mathbb{Z}_6$

$1 \to 0$
$2 \to 1$
$4 \to 2$
$8 \to 3$
$7 \to 4$
$5 \to 5$

is an isomorphism.

(11.) $G = \{2^n : n \in \mathbb{N}\}$

Show $(G, \cdot)$ satisfies G1, G2 and G3.

Proof. We will show $(G, \cdot)$ is associative (G1).

Let $a, b, c \in G$. Note

$(ab)c = a(bc)$ since $a, b, c \in \mathbb{R}$ and mult. over $\mathbb{R}$ is associative.

The multiplicative identity is $e = 1$ and $1 \in G$ since $1 = 2^n$ where $n = 0$. So $(G, \cdot)$ satisfies G2.

Let $a \in G$. So $a = 2^n$ for some $n \in \mathbb{Z}$. Note $-n \in \mathbb{Z}$ so $2^{-n} \in G$. Note $a \cdot 2^{-n} = 2^n \cdot 2^{-n} = 2^0 = 1$, and

$2^{-n} \cdot a = \cdots = 1$.

So $(G, \cdot)$ is closed inverse (ie $(G, \cdot)$ satisfies G3). $\square$

(12.) Hint: Let $f : \mathbb{Z} \to G$ be defined by

$f(n) = 2^n$. You must show

(1) f is a bijection, and

(2) $f(n+m) = f(n) \cdot f(m) \quad \forall\, n, m \in \mathbb{Z}$.