

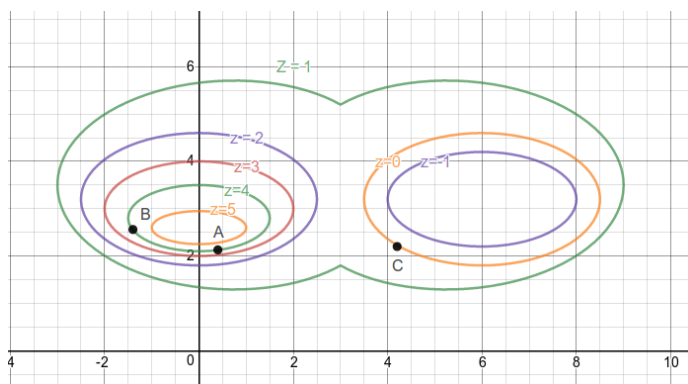
## 1 Paths

1. Let  $\mathbf{r}(t) = \langle t^2 - t, t^3 - 3t^2 + 3 \rangle$ 
  - (a) Find the position, velocity and acceleration of the particle at time  $t = 2$ .
  - (b) Graph the position, velocity and acceleration appropriately.
2. Let  $\mathbf{r}(t) = \langle \sin(e^{-t}), \cos(e^{-t}) \rangle$ 
  - (a) Find the speed function
$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$
  - (b) Compute the arc length from  $t = 0$  to  $t = 1$ .
  - (c) Compute the arc length from  $t = 0$  to  $t = \infty$ .
  - (d) What is the graph of  $\mathbf{r}(t)$ ?
3. Let  $\mathbf{r}(t) = \langle e^{-t} \sin(t), e^{-t} \cos(t) \rangle$ 
  - (a) Find the speed function
$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$
  - (b) Compute the arc length from  $t = 0$  to  $t = 1$ .
  - (c) Compute the arc length from  $t = 0$  to  $t = \infty$ .
  - (d) What is the graph of  $\mathbf{r}(t)$ ?

## 2 Functions of Several Variables

4. Compute the following limits if they exist. If not show why.
  - (a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2 + 1}$
  - (b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos(x^2 + y^2)}{x^2 + y^2}$
  - (c)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + xy + y^2}{x^2 + y^2}$
5. Let  $f(x, y) = x^2 + y^2$ . Consider the points  $P(0, 2)$  and  $Q(1, 2)$ .
  - (a) Graph the contour plot. Include  $z = -1, 0, 1, 2, 3, 4$ .
  - (b) Compute the  $\nabla f(x, y)$
  - (c) Compute the  $\nabla f(P)$  and  $\nabla f(Q)$ . Also compute their norms.

- (d) Graph  $\nabla f(P)$  and  $\nabla f(Q)$  with initial points  $P$  and  $Q$  respectively.
6. Draw the gradient at each point  $A$ ,  $B$  and  $C$ .



7. Fill in the blanks.
- The gradient is the direction of \_\_\_\_\_ increase.
  - The gradient is \_\_\_\_\_ to the contour lines.
  - The norm of the gradient is \_\_\_\_\_.
  - A larger norm of one gradient is \_\_\_\_\_ graphically.
8. Let  $f(x, y) = e^{xy^2+2} - xy^3 + 2$ . Find the tangent plane to  $f(x, y, z)$  at the point  $(-2, 1)$ . Use that plane to estimate  $f(-2.1, 0.8)$ . Compare to the real value of  $f(-2.1, 0.8)$ .
9. Let  $f(x, y, z) = x^2y - xy^2 + z^3$ . Find the tangent plane (actually a hyperplane) to  $f(x, y, z)$  at the point  $(1, 2, 3)$ .
10. Let  $A(1, 2, 3)$  and  $B(1, 1, 2)$  be points in  $\mathbb{R}^3$ . Let  $\mathbf{v} = \langle 1, 2, 3 \rangle$  and  $\mathbf{w} = \langle 2, 2, 4 \rangle$  and  $f(x, y, z) = x^2 + y^2 + z^2$ .
- Compute  $\nabla f(A)$  and  $\|\nabla f(A)\|$ .
  - Compute  $D_{\mathbf{v}}f(B)$  and  $D_{\mathbf{w}}f(B)$ . One is bigger than the other. Interpret.
  - Compute  $D_{\mathbf{v}}f(A)$ . Compare to  $\|\nabla f(A)\|$ .
11. Let  $f(x, y) = x^3 + x^2y^2$ . Let  $x(t) = t^2 + 1$  and  $y(t) = 2t - 1$ .
- Use the chain rule to compute  $\frac{df}{dt}$ .
  - Compute  $\frac{df}{dt}$  at  $t = 1$ .
  - Compute the point given in the path at  $t = 1$ .
  - Let  $\mathbf{v} = \langle 2, 2 \rangle$ . Compute  $D_{\mathbf{v}}f(2, 1)$ . Is this question familiar to you?
12. Find and classify extrema.
- $f(x, y) = x^2 - xy + y^3$ .

- (b)  $f(x, y) = x^2 + 2xy - y^4$ .  
(c)  $f(x, y, z) = x + 3y + z$  subject to  $x^2 + y^2 + z^2 = 1$ .  
(d)  $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$  subject to  $x + y + z - 3w = 4$ .  
(e)  $f(x, y, z, w) = x \ln(x) + y \ln(y) + z \ln(z)$  subject to  $x + y + z = 1$ . In this problem  $f$  is called information entropy.

### 3 Integrals

13.  $\iint_R x + y \, dA$  over the region defined by  $x + y = 2$  and the coordinate axes.
14.  $\iint_R xy \, dA$  over the region defined by  $y = x^2$  and the line  $y = x + 1$ .
15.  $\iint_R e^{x^2} \, dA$  over the region defined by  $y = -x$ ,  $y = 2x$  and the vertical line  $x = 4$ .
16.  $\iint_R e^{x^2+y^2} \, dA$  over the region defined by the portion of the circle  $x^2 + y^2 = 4$  in the third quadrant.
17.  $\iint_R \sqrt{\frac{\tan^{-1}(y/x)}{x^2 + y^2}} \, dA$  over the region defined by the portion of the circle  $x^2 + y^2 = 4$  above the lines  $y = -x$  and  $y = x$ .
18. Find the volume below the paraboloid  $z = 12 - x^2 - y^2$  and above the  $xy$ -plane.
19.  $\iint_R \sin(x - y) \cos(x + y) \, dA$  over the region defined the lines  $y = x + 2$ ,  $y = x + 4$ ,  $y = -x$  and  $y = -x + 3$ . Hint the change of variables is  $u = x - y$  and  $v = x + y$ .
20.  $\iint_R \frac{x - y}{2x + y} \, dA$  over the region defined the lines  $y = x + 2$ ,  $y = x$ ,  $y = -2x + 2$  and  $y = -2x + 3$ .
21.  $\iint_R xy \, dA$  over the region defined the graphs of  $xy = 1$ ,  $xy = 3$  and the lines  $y = x$  and  $y = 3x$ . Hint  $x = u/v$  and  $y = v$ .
22.  $\iint_R (x - y)e^{x^2-y^2} \, dA$  over the region defined the lines  $y = x + 2$ ,  $y = x$ ,  $y = -x$  and  $y = -x + 3$ .
23.  $\iint_R e^{x^2+4y^2} \, dA$  over the region defined by the portion of the ellipse  $\frac{x^2}{4} + y^2 = 1$  in the third quadrant. Hint use the change of variables  $x = 2v \cos(u)$  and  $x = v \sin(u)$ . And note I had  $\pi \leq u \leq \frac{3\pi}{2}$

## 4 Line Integrals

24.  $\int_C x \, dx$ . Let  $C$  be line segment from  $(0, 1)$  to  $(3, 2)$ .
25.  $\int_C xy \, ds$ . Let  $C$  be line segment from  $(0, 1)$  to  $(3, 2)$ .
26.  $\int_C \langle -x, y \rangle \cdot d\mathbf{r}$ . Let  $C$  be line segment from  $(0, 1)$  to  $(3, 2)$ .
27.  $\int_C x \, dy$ . Let  $C$  be line segment from  $(0, 1)$  to  $(3, 2)$ .
28.  $\oint_C xy \, dx$ . Let  $C$  be outside of the triangle traced from  $(0, 0)$  to  $(0, 2)$  to  $(1, 2)$  and then back to  $(0, 0)$ .
29.  $\oint_C \langle -x, y \rangle \cdot d\mathbf{r}$ . Let  $C$  be outside of the triangle traced from  $(0, 0)$  to  $(0, 2)$  to  $(1, 2)$  and then back to  $(0, 0)$ .
30.  $\oint_C \langle 1, xy \rangle \cdot d\mathbf{r}$ . Let  $C$  be the circle  $x^2 + y^2 = 4$  traced counter-clockwise.
31.  $\oint_C -x + y \, ds$ . Let  $C$  be the circle  $x^2 + y^2 = 4$  traced counter-clockwise.

## 5 Green's Theorem

32.  $\oint_C \langle x, -y \rangle \cdot d\mathbf{r}$ . Let  $C$  be outside of the square traced from  $(0, 0)$  to  $(0, 2)$  to  $(1, 2)$  to  $(1, 0)$  and then back to  $(0, 0)$ .
33.  $\oint_C \langle e^{x^3} - xy, e^{y^3} - y \rangle \cdot d\mathbf{r}$ . Let  $C$  be outside of the triangle traced from  $(0, 0)$  to  $(0, 2)$  to  $(1, 2)$  and then back to  $(0, 0)$ .
34.  $\oint_C \langle \cos(x^2) + y, \cos(y^2) + xy \rangle \cdot d\mathbf{r}$ . Let  $C$  be the circle  $x^2 + y^2 = 4$  traced counter-clockwise.

## 6 Div/Grad/Curl

35. Define

$$f(x, y, z) = x^3 - yz^2 \text{ and } \mathbf{F}(x, y, z) = \langle x^3, yz^2, xy \rangle.$$

Compute the following, if possible, and if not possible state why.

- (a)  $\operatorname{div}(f(x, y, z))$
- (b)  $\operatorname{grad}(f(x, y, z))$
- (c)  $\operatorname{curl}(f(x, y, z))$
- (d)  $\operatorname{div}(\mathbf{F}(x, y, z))$
- (e)  $\operatorname{grad}(\mathbf{F}(x, y, z))$
- (f)  $\operatorname{curl}(\mathbf{F}(x, y, z))$
- (g)  $\nabla \cdot \mathbf{F}(x, y, z)$
- (h)  $\nabla \times (\nabla \cdot \mathbf{F}(x, y, z))$
- (i)  $\nabla \times (\nabla f(x, y, z))$