Math 6250: Final Exam Review

1 Calculus

1. Prove $\lim_{n\to\infty} \frac{n^2+1}{3n^2+2} = \frac{1}{3}$
2. Prove $\lim_{n\to\infty} \frac{n^2+1}{3n+2} = \infty$
3. Assume $\lim_{n\to\infty} a_n = a$. Show (a_n) is bounded.
4. Assume (a_n) and (b_n) are sequences in $\mathbb{R}$. Show the following implication is **not** true
   If $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\in\infty} a_n b_n = 0$.
5. Assume (a_n) and (b_n) are sequences in $\mathbb{R}$. Show the following implication **is** true
   If $\lim_{n\to\infty} a_n = 0$ and (b_n) is bounded then $\lim_{n\in\infty} a_n b_n = 0$.
6. Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Prove
   (a) $\lim_{n\to\infty} k = 0$ where $k \in \mathbb{R}$.
   (b) $\lim_{n\to\infty} ka_n = ka$ where $k \in \mathbb{R}$.
   (c) $\lim_{n\to\infty} a_n + b_n = a + b$
   (d) $\lim_{n\to\infty} a_n b_n = ab$
7. Prove $\lim_{x\to 3} 3x - 5 = 4$
8. Prove $\lim_{x\to 2} 3x^2 - 5 = 7$
9. Assume $\lim_{x\to a} f(x) = F$. Prove $f(x)$ is bounded near $x = a$. That is, show there is a $M \in \mathbb{R}$ and a $\delta > 0$ so that
   If x in Domain and $0 < |x - a| < \delta$ then $|f(x)| < M$
10. Assume $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$. Prove
    (a) $\lim_{x\to a} k = k$ where $k \in \mathbb{R}$.
    (b) $\lim_{x\to a} kf(x) = kF$ where $k \in \mathbb{R}$.
    (c) $\lim_{x\to a} f(x) + g(x) = F + G$
    (d) $\lim_{x\to a} f(x)g(x) = FG$
11. If $f(x)$ and $g(x)$ are differentiable at $x = a$. Prove the following:

(a) The derivative of a constant is zero.
(b) $[kf(x)]' = kf'(x)$ where $k \in \mathbb{R}$
(c) $[f(x) + g(x)]' = f'(x) + g'(x)$
(d) $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
(e) $[x^n]' = nx^{n-1}$ for all $n \in \mathbb{N}$

12. If $f(x)$ is differentiable at $x = a$ then f is continuous at $x = a$.

13. Show the inequality that $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}$ for $\theta \in (0, \pi/2)$.

14. Use the above inequality to show $[\sin(x)]' = \cos(x)$. Hint use $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

15. Use the $[\sin(x)]' = \cos(x)$ to find the derivatives for the other five trigonometric functions.

16. Compute the various derivatives of the new functions we learned involving $\sin(1/x)$ and absolute values.

2 Area and Dimension

17. Describe clearly a fractal of dimensions of $\frac{\ln(4)}{\ln(3)}$ and $\frac{\ln(5)}{\ln(3)}$. Describe the Koch snowflake.

18. Define the area of the unit circle as π. Use this definition and what we did in class to show that the area of a circle of radius r is πr^2, the circumference of a circle is $2\pi r$ and the volume of a sphere of radius r is $\frac{4}{3}\pi r^2$.

19. Compute $\Gamma(1)$, $\Gamma(2)$, $\Gamma(3)$ and be able to prove the recursion formula $\Gamma(n) = (n-1)\Gamma(n-1)$.

20. Prove the three mean inequalities for arithmetic, geometric and harmonic.

21. Use the arithmetic-geometric mean inequality to prove

    for all $n \in \mathbb{N}$ with $n > 1$

$$\sqrt[n]{1 + \frac{\sqrt[n]{n}}{n}} + \sqrt[n]{1 - \frac{\sqrt[n]{n}}{n}} < 2.$$