Math 6250 Quiz 1

Name:____

1. Show the following for all $n \in \mathbb{N}$. Use induction.

$$1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

   Note this can be shown geometrically. See if you can prove this by just drawing a few squares.

2. Recall the binomial theorem:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + b^n\end{aligned}$$

   where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

   (a) Prove the Lemma

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

   (b) Prove the binomial theorem for $n \in \mathbb{N}$ using induction and the above Lemma.

3. Let $a, b \in \mathbb{Z}$. Define the following relation.

$$a\mathcal{R}b \Leftrightarrow a - b \text{ is divisible by } 5$$

   Show $\mathcal{R}$ is an equivalence relation.

4. Let $a, b \in \mathbb{Z}$. Define the following relation.

$$a\mathcal{R}b \Leftrightarrow a - b = 5$$

   Show $\mathcal{R}$ is not an equivalence relation. In fact show $\mathcal{R}$ is not symmetric, is not reflexive and is not transitive. Find a counter example for each.

5. Here we use an equivalence relation on $\mathbb{N}$ to define the integers. Let $a, b \in \mathbb{N}$. Define the following relation.

$$(a,b)\mathcal{R}(a',b') \Leftrightarrow a + b' = a' + b$$

And define multiplication as

$$(a,b) + (c,d) = (a+c, b+d)$$

$$(a,b) \cdot (c,d) = (ac + bd, ad + bc)$$

   (a) Show $\mathcal{R}$ is an equivalence relation.
   (b) Show addition is well defined. That is, Show
   If $(a,b)\mathcal{R}(a',b')$ and $(c,d)\mathcal{R}(c',d')$ then $(a,b)+(c,d)\mathcal{R}(a',b')+(c',d')$.
   (c) Show multiplication is well defined.

6. Write the definition for the equivalence relation on $\mathbb{Z}$ to define $\mathbb{Q}$. Show it is an equivalence relation and show that multiplication is well defined.

7. Show if $x^2 = 5$ then x is not a Rational number.