Math 4160 - Practice Test 2

1. Set of problems like in class for inner products. eg prove an inner product is additive in the econd position, prove an inner product is conjugate homogenious in the second coordinate...

2. Prove $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ in a vector space over $\mathbb{R}$.

3. Let e_1, e_2, e_3 be an orthonormal basis for an inner product space V. Prove
$$\|a_1e_1 + a_2e_2 + a_3e_3\| = \sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2}.$$

4. Let $u, v \in V$ where V is a real inner product space. Prove u is othogonal to w where w is defined as
$$w = v - \frac{\langle u, v\rangle}{\|u\|^2}u$$

5. Assume $\|Tv\| \leq \|v\|$ for all $v \in V$ an inner product space. Show $T - \sqrt{2}I$ is invertible.

   Hint: Note λ is an eigenvalue iff $T - \lambda I$ is not invertible.

6. Let $u, v \in V$ where V is an inner product space. Then $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbb{F}$ if and only if $\|u\| = \|v\|$.

7. Let $u_1 = (1, 2, 3), u_2 = (1, 1, 3)$ and $u_3 = (0, -1, 2) \in \mathbb{R}^3$. Use Gram-Schmidt Orthoganilzation to find an o-n basis.

8. Let $p_1 = 1, p_2 = x$ and $p_3 = x^2 \in \mathcal{P}_2(\mathbb{R})$. Use Gram-Schmidt Orthoganilzation to find an o-n basis. Use the inner product below
$$\langle f, g\rangle = \int_0^1 fg dx.$$

9. Find $U^\perp$ for the following sets U

   (a) $U = \{(1, 0, 1)\} \subseteq \mathbb{R}^3$ with the usual inner product.

   (b) $U = \{1\} \subseteq \mathcal{P}_2(\mathbb{R})$ with the inner product below
$$\langle f, g\rangle = \int_0^1 fg dx.$$

10. Find eigenvalues and cooresponding eigenvectors for the following operators

    (a) $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ where $T(p) = p'$

    (b) $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ where $T(p) = xp'$

11. Let $V = U \oplus W$. Assume U and W are nonzero subspaces of V. And define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ where $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

12. Let $T \in \mathcal{L}(V)$ and let $v, w \in V$ be nonzero vectors so that

$$T(u) = 3w \text{ and } T(w) = 3u.$$

    Prove that either 3 or -3 is an eigenvalue of T.

13. Let $T \in \mathcal{L}(V)$ and assume $T^n = 0$ where $n \in \mathbb{N}$.

    (a) Show $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + T^2 + \cdots + T^{n-1}$$

    (b) Note $I - 2^n T$ is invertible also. what is $(I - 2^n T)^{-1}$?

    (c) What if we know that $I - T^3 = 0$. Then guess T^{-1}

14. Prove $\{v_1, v_2, \ldots, v_m\}^\perp = [span\{v_1, v_2, \ldots, v_m\}]^\perp$

15. Define **Projection** and $U^\perp$.

16. Let $T \in \mathcal{L}(V)$ so that

    - $T^2 v = Tv$ for all $v \in V$ and
    - every vector in Null(T) is orthogonal to every vector in the Range(T) .

    Show that T is a projection on some set U.