Math 4160 - Test 1 Review

1 Linear Spaces

1. Definition of a **Vector Space**.
2. Prove $\mathbb{R}^3$ with the usual vector addition and scalar multiplication is a vector space.
3. Prove $\mathbb{F}^\infty$ with the usual vector addition and scalar multiplication is a vector space.
4. Prove the additive identity is unique.
5. Prove $(-1)\mathbf{v} = -\mathbf{v}$.
6. Definition of a **subspace**.
7. State the subspace test.
8. Prove the following are subspaces of $V = \mathbb{R}^3$ or are not subspaces.
   (a) $\{(x, y, z)|x + y - z = 0\}$
   (b) $\{(x, y, z)|xyz = 1\}$
   (c) $\{(x, y, z)|x + y - z = 1\}$
   (d) $\{(x, y, z)|x + y - z = 0 \text{ and } 2x = z\}$
9. Let U and W be suspaces of V. Prove $U + W$ is a subspace of V.
10. Let U and W be suspaces of V. Prove $U \cap W$ is a subspace of V.
11. Let U and W be suspaces of V. Prove $U \cup W$ is **not** a subspace of V.

2 Span, Bases and Dimension

12. The definition of a **linear combination**, **linear independence**, **Basis**, **Dimension** and **Finite Dimensional Vector Space**.
13. Answer yes or no and justify your answer.
    (a) Is $(1, 2, 3) \in \mathrm{Span}((1, -2, 1), (1, 1, 1))$?
    (b) Is $x^2 - 1 \in \mathrm{Span}(x, x - x^2, x + x^2)$?

(c) Is $(1,2,3) \in \mathrm{Span}((1,-2,1),(3,-12,2))$?

(d) Is $(1,2,3) \in \mathrm{Span}((4,5,6),(7,8,9))$?

14. Find a a basis and dimension for the following.

    (a) $\mathrm{Span}((1,-2,1),(1,1,1),(-2,3,-2))$

    (b) $\{p \in \mathcal{P}(\mathbb{R}) : p'(3) = 0\}$

    (c) $\{p \in \mathcal{P}_3(\mathbb{R}) : p'(3) = 0\}$

    (d) $\{(x,y,z,w) \in \mathbb{R}^4 : x + y - 2z = 0 \text{ and } 3y - z - w = 0\}$

15. Are the following lists linearlly independent? Justify your answer.

    (a) $\mathrm{Span}((1,-2,1),(1,1,1),(-2,3,-2))$

    (b) $\{x, x - x^2, x + x^2\}$

    (c) $\{(1,-2,1),(3,-12,2),(1,2,3)\}$

16. Prove: If list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is linearlly dependent then one of the vectors is a linear combination of the other two vectors.

17. Prove: If for the list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ one of the vectors is a linear combination of the other two vectors then the list is linearlly dependent.

3 Linear Maps, Null Spaces, Range Spaces

18. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V,W)$.

    (a) If rank(T) = 2 and the dim(V) = 3 what can Nullity(T) be?

    (b) If rank(T) = 2 and the T is surjective what is the dim(W)?

    (c) If rank(T) = 2 and the T is injective what is the dim(V)? The Nullity(T)? What are the possible values for dim(W)?

    (d) If rank(T) = 2 and dim(V) = dim(W) =4, what is the Nullity(T')? What is the Rank(T')?

    (e) If rank(T) = 1 and dim(V) = dim(W) =4, what is the Nullity(T')? What is the Rank(T')?

19. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V,W)$. Prove if T is injective then Nullity(T) = 0.

20. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V, W)$. Prove if Nullity(T) = 0 then T is injective.

21. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V, W)$. Prove if T is surjective then Rank(T) = dim(W).

22. Suppose V and W are finite dimensional vector spaces. Prove there is some $T \in \mathcal{L}(V, W)$ that is injective if and only if $dim(V) \leq dim(W)$.

4 Linear Maps as Matrices, Isomporphisms

23. Show the spaces $\mathcal{P}_2(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$.

24. Show the spaces $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ where both are considered as vector spaces over $\mathbb{R}$.

25. Define $V = \{(x, y, z, w) : x+y = 0 \text{ and } x-z-w = 0\}$ and let $W = \mathbb{R}^2$. Find an isomorphism from V to W.

5 Duality

26. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ where $T(x, y) = (x, y, x + y)$. Compute

    (a) $T'(\phi)(1, 2)$ where $\phi = (1, -3, 5)$
    (b) $T'(\phi)(u, v)$ where $\phi = (1, -3, 5)$
    (c) $T'(\phi)(u, v)$ where $\phi = (a, b, c)$
    (d) Null(T)
    (e) Range(T)
    (f) Null(T')
    (g) Range(T')
    (h) Range$(T)^\circ$
    (i) Compute the dimension of each answer above.

27. Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ where $T(x_1, x_2, x_3, x_4) = (x_1, x_2 - 2x_4, 3x_2 + x_4)$. Compute

    (a) $T'(\phi)(1, 2, 3, 4)$ where $\phi = (1, -1, 2)$
    (b) $T'(\phi)(u_1, u_2, u_3, u_4)$ where $\phi = (1, -1, 2)$

(c) $T'(\phi)(u_1, u_2, u_3, u_4)$ where $\phi = (a, b, c)$

(d) Null(T)

(e) Range(T)

(f) Null(T')

(g) Range(T')

(h) Range$(T)^\circ$

(i) Compute the dimension of each answer above.

28. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V, W)$. Prove if T is injective then T' is surjective.

29. Suppose V and W are finite dimensional vector spaces. Assume $T \in \mathcal{L}(V, W)$. Prove if T is surjective then T' is injective.

30. Prove that Range$(T)^\circ$ = Null(T').

not on test but should have been

1. Define $D : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ as $D(p) = 3p'' - p'$.

   (a) Find the matrix for D using the standard bases.

   (b) Find the matrix for D using the standard bases $B = \{1 - x^2, 1 + x^2, x^3 + x, x^3 - x\}$ for the domain and the standard basis for the codomain.

   (c) Find the matrix for D using the standard bases $B = \{1 - x^2, 1 + x^2, x^3 + x, x^3 - x\}$ for the domain and $B = \{1, 1 + x^2, x^2, x^3\}$ for the codomain.

2. For the basis $B = \{1 - x^2, 1 + x^2, x^3 + x, x^3 - x\}$ for $D : \mathcal{P}_3(\mathbb{R})$ what is its dual basis?