Math 3160 - Final Exam

Name:____

No calculators and show all work.

1. Find the solution to the given linear system.

$$\begin{cases} x_1 & +2x_2 & +3x_3 & = 4 \\ x_1 & +x_2 & +x_3 & = 0 \\ -x_1 & & -x_3 & = 6 \end{cases}$$

2. Finish the following definieions

   - A **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is

   - A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearally independent** if

   - A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if and only if

3. Let $P(1,0,4)$, $Q(0,3,0)$ and $R(0,3,4)$ be points in $\mathbb{R}^3$.

   (a) Compute the area of the triangle formed by the points P, Q and R.

   (b) What is the standard equation of the plane containing the triangle from Problem 3a?

   (c) What is the parametric (or vector) equation of the plane containing the triangle from Problem 3a?

4. Let $V = \{(x, y) \in \mathbb{R}^2 | y \neq 0\}$. And define the two operations

$\oplus$: $(x_1, y_1) \oplus (x_2, y_2) = (2x_2 + x_1y_2 - x_2y_1, y_1y_2)$

$\odot$: $k \odot (x_1, y_1) = (kx_1, y_1)$

(a) Compute $(1, 3) \oplus (2, 5)$.

(b) Show V is **not** commutative.

(c) Note V has a zero vector. Prove $(0, 1)$ is the zero vector for V.

5. The linear transformation is given by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_3 \\ x_2 \\ -x_1 + 3x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ 2x_2 \end{bmatrix}.$$

(a) Find the matrix, A, to represent the linear trasformation T.

(b) Compute the basis for the Range of T.

(c) Find a basis for the null space of A, NULL(A).

(d) Find the rank and nullity.

(e) What is the dimension of the domain of T and the codomain of T? Compare Rank, Nullity and the dimension of the Domain.

6. Let a sequence be defined by the following recursive formula

$$a_1 = 1, a_2 = 1 \text{ and } a_{n+2} = 3a_{n+1} + 4a_n$$

(a) Compute the first five terms of the sequence.

(b) Find a matrix A for the sequence as we did in class.

(c) Diagonalize A. That is find D and P so that $A = PDP^{-1}$.

(d) Compute $A^n \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ using Problem 6c.

7. Define the follwing matrix: $E = \begin{bmatrix} 5 & 8 \\ 1 & 7 \end{bmatrix}$

(a) For the plaintext message "METS", find the two letter block representation using the alphabet: $A = 00, B = 01, C = 02, D = 03, \ldots$

(b) Use the matrix E to encipher the plaintmessage.

8. Define the follwing matrix: $E = \begin{bmatrix} 5 & 8 \\ 1 & 7 \end{bmatrix}$

   (a) Find the deciphering matrix D.

   (b) Decipher the ciphertext: PACW.

9. Find the steady states for the following Markov process (if regular) with the given transition matrix

$$A = \begin{bmatrix} .1 & .5 \\ .9 & .5 \end{bmatrix}$$

10. Prove the following:

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ be a linearly dependent subset of the vector space V. Say

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n = 0$$

where a_1 is not zero.

Show that one of the elements of S is a lenear combination of the other elements of S.

11. Asssume $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^n$ satisfy the following properties

    - $\mathbf{u}_1 \cdot \mathbf{u}_1 = 1$, $\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$ and $\mathbf{u}_3 \cdot \mathbf{u}_3 = 1$
    - $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_3 \cdot \mathbf{u}_2 = 0$

    Show the change of basis matrix $P_{\text{STANDARD} \to B}$ is given by the matrix with rows $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.