MATH 5320 Test 1: Practice

1. Know the definitions

(1) $|A| = |B|$ or $A \sim B$
(2) $f : A \to B$ is injective.
(3) $f : A \to B$ is onto.
(4) $f : A \to B$ is bijective.
(5) upper bound, lower bound, sup, inf
(6) $a_n \to a$ or $a_n \to \infty$

2. Some good facts to know

(1) the compteteness axiom
(2) $\mathbb{N}$ is unbounded
(3) Squeeze in
(4) Archimedean Property
(5) the density of $\mathbb{Q}$ in $\mathbb{R}$
(6) Weieretrass (Monotone Convergent Theroem)

3. Cardinality

(1) Let A, B and C be disjoint sets. Show if $A \sim B$ and $B \sim C$ then $A \sim C$.
(2) Let A, B and C be disjoint sets. Show if $A \sim B$ then $A \cup C \sim B \cup C$.
(3) Let A_n be a countable set for each $n \in \mathbb{N}$. Prove $\cup A_n$ is countable.
(4) Show the sets $\{\frac{1}{n} : n \in \mathbb{N}\} \sim \{\frac{1}{n+1} : n \in \mathbb{N}\}$.
(5) Show the sets $[a, b] \sim [0, 1]$ where $a < b$.
(6) Show the sets $(0, \infty) \sim (0, 1)$.
(7) Show the sets $(-\infty, \infty) \sim (0, 1)$.
(8) Show the sets $\mathbb{N} \sim \mathbb{Z}$.
(9) Show the sets $\mathbb{N} \sim \mathbb{Q}$.
(10) Show the sets $\mathbb{N} \not\sim \mathbb{R}$.

4. Functions and Suprema

(1) Let the set A be nonempty and bounded above and let $\alpha = \sup(A)$. Show for all $\varepsilon > 0$ there is some $x \in A$ so that $\alpha - \varepsilon < x \leq \alpha$.
(2) Let the set A be nonempty and bounded above and define $B = 3A = \{3a : a \in A\}$. Prove $\sup(B) = 3\sup(A)$
(3) Either prove or disprove that the functions defined below are injective or surjective.
 (a) Let $f : (0, \infty) \to (0, \infty)$, where $f(x) = x^2$.
 (b) Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$.
 (c) Let $f : A \to A$, where $f(x) = x$.
(4) Let $f : S \to T$ and let $g : T \to U$ then
 (a) Show if f is injective and g is injective then $g \circ f$ is injective.
 (b) Show if f is surjective and g is surjective then $g \circ f$ is surjective.
 (c) Show if $g \circ f$ is injective then f is injective.
 (d) Show if $g \circ f$ is surjective then g is surjective.
 (e) Show by couterexample that if $g \circ f$ is injective and f is injective then it is not necessary that g is injective.
 (f) Show by couterexample that if $g \circ f$ is surjective and g is surjective then it is not necssary that f is surjective.

5. Sequences

(1) Prove using the definition that
 (a) $\lim_{n\to\infty} \dfrac{k}{n} = 0$ for any $k \in \mathbb{R}$
 (b) $\lim_{n\to\infty} \dfrac{3n+1}{n+2} = 3$
(2) Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Show $\lim_{n\to\infty} a_n b_n = ab$.
(3) Assume $\lim_{n\to\infty} a_n = 0$ and and assume the sequence (b_n) is bounded. Show

$$\lim_{n\to\infty} a_n b_n = 0.$$

(4) Assume $\lim_{n\to\infty} a_n = 0$ and and assume the sequence (b_n) is not bounded. Show $\lim_{n\to\infty} a_n b_n$ is not necessarily zero. That is find (a_n) and (b_n) where $a_n \to 0$ but $a_n b_n \not\to 0$.

(5) Prove If (a_n) is convergent then (a_n) is bounded.

(6) Know If (a_n) is monotone and bounded then (a_n) is convergent. (This is called the Monotone Convergence Theorem)

(7) State and prove the Squeeze Theorem.

(8) Use the Monotone Convergence Theorem to show (a_n) as described below has a limit. Compute that limit.
   (a) $a_1 = 1, a_{n+1} = \frac{a_n+1}{a_n+2}$
   (b) $a_1 = 1, a_{n+1} = \sqrt{a_n + 1}$

(9) Show the following sequences diverge to infinity.
   (a) $\lim_{n\to\infty} 3n - 1 = \infty$
   (b) $\lim_{n\to\infty} \frac{n+5}{\sqrt{n+1}} = \infty$
   (c) $\lim_{n\to\infty} a_n = \infty$ where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.