MATH 5320 Test 1: Practice

1 Know the definitions and Theorems

1. $|A| = |B|$ or $A \sim B$
2. $f : A \to B$ is injective.
3. $f : A \to B$ is onto.
4. $f : A \to B$ is bijective.
5. upper bound, lower bound, sup, inf
6. $a_n \to a$ or $a_n \to \infty$
7. (a_n) is Cauchy
8. $\lim_{x \to c} f(x) = L$
9. $f : A \to B$ is continuous at $x = c$
10. $f : A \to B$ is continuous
11. $f : A \to B$ is uniformly continuous
12. $f : A \to B$ is linear
13. Some good theorems to know
    (a) Bolzano-Weierstrass Theorem
    (b) Extreme Value Theorem
    (c) Intermediate Value Theorem
    (d) Monotone convergence theorem.
    (e) Sequential criterion for limits

2 Functions and Suprema

14. Let the set A be nonempty and bounded above and let $\alpha = \sup(A)$. Show for all $\varepsilon > 0$ there is some $x \in A$ so that $\alpha - \varepsilon < x \leq \alpha$.
15. Let the set A be nonempty and bounded above and define $B = 3A = \{3a : a \in A\}$. Prove $\sup(B) = 3\sup(A)$

16. Either prove or disprove that the functions defined below are injective or surjective.

    (a) Let $f : (0, \infty) \to (0, \infty)$, where $f(x) = x^2$.

    (b) Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$.

    (c) Let $f : A \to A$, where $f(x) = x$.

17. Let $f : S \to T$ and let $g : T \to U$ then

    (a) Show if f is injective and g is injective then $g \circ f$ is injective.

    (b) Show if f is surjective and g is surjective then $g \circ f$ is surjective.

    (c) Show if $g \circ f$ is injective then f is injective.

    (d) Show if $g \circ f$ is surjective then g is surjective.

    (e) Show by couterexample that if $g \circ f$ is injective and f is injective then it is not necessary that g is injective.

    (f) Show by couterexample that if $g \circ f$ is surjective and g is surjective then it is not necssary that f is surjective.

3 Cardinality

18. Let A, B and C be disjoint sets. Show if $A \sim B$ and $B \sim C$ then $A \sim C$.

19. Let A, B and C be disjoint sets. Show if $A \sim B$ then $A \cup C \sim B \cup C$.

20. Show the sets $\{\frac{1}{n} : n \in \mathbb{N}\} \sim \{\frac{1}{n+1} : n \in \mathbb{N}\}$.

21. Show the sets $[a, b] \sim [0, 1]$ where $a < b$.

22. Show the sets $(0, \infty) \sim (0, 1)$.

23. Show the sets $(-\infty, \infty) \sim (0, 1)$.

24. Show the sets $\mathbb{N} \sim \mathbb{Z}$.

25. Show the sets $\mathbb{N} \sim \mathbb{Q}$.

26. Show the sets $\mathbb{N} \not\sim \mathbb{R}$.

4 Sequences

27. Prove using the definition that

    (a) $\lim_{n \to \infty} \dfrac{k}{n} = 0$ for any $k \in \mathbb{R}$

    (b) $\lim_{n \to \infty} \dfrac{3n+1}{n+2} = 3$

28. Assume $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$. Show $\lim_{n \to \infty} a_n b_n = ab$.

29. Assume $\lim_{n \to \infty} a_n = 0$ and and assume the sequence (b_n) is bounded. Show
$$\lim_{n \to \infty} a_n b_n = 0.$$

30. Assume $\lim_{n \to \infty} a_n = 0$ and and assume the sequence (b_n) is not bounded. Show $\lim_{n \to \infty} a_n b_n$ is not necessarily zero. That is find (a_n) and (b_n) where $a_n \to 0$ but $a_n b_n \not\to 0$.

31. Prove If (a_n) is convergent then (a_n) is bounded.

32. Prove If (a_n) is monotone and bounded then (a_n) is convergent. (This is called the Monotone Convergence Theorem)

33. Prove the Squeeze Theorem.

34. Use the Monotone Convergence Theorem to show (a_n) as described below has a limit. Compute that limit.

    (a) $a_1 = 1, a_{n+1} = \frac{a_n+1}{a_n+2}$

    (b) $a_1 = 1, a_{n+1} = \sqrt{a_n + 1}$

35. Show the following sequences diverge to infinity.

    (a) $\lim_{n \to \infty} 3n - 1 = \infty$

    (b) $\lim_{n \to \infty} \frac{n+5}{\sqrt{n+1}} = \infty$

    (c) $\lim_{n \to \infty} a_n = \infty$ where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.

5 Limits of Functions

36. prove using the $\varepsilon - \delta$ definition of a limit

$$\lim_{x \to -2} 3x - 1 = -7 \text{ and } \lim_{x \to 3} x^3 - 8 = 19 \text{ and } \lim_{x \to -2} \frac{1}{1 + x^2} = \frac{1}{5}$$

$$\lim_{x \to 3} x^2 - 2x = 3 \text{ and } \lim_{x \to 1} \sqrt{x + 3} = 2$$

37. If $\lim_{x \to c} f(x) = F$ and $\lim_{x \to c} g(x) = G$ then show $\lim_{x \to c} f(x)g(x) = FG$.

38. If $\lim_{x \to c} f(x) = F$ and $\lim_{x \to c} g(x) = G$ then show $\lim_{x \to c} f(x) + g(x) = F + G$.

39. If $\lim_{x \to c} f(x) = F$ and let $k \in \mathbb{R}$ then show $\lim_{x \to c} kf(x) = kF$.