Math 3520 - Test 2 Review

1 Cardinality

1. Show the sets have the same cardinality
   (a) $\mathbb{N}$ and $\mathbb{Z}$
   (b) $\{a, b, c\}$ and $\{1, 2, 7\}$
   (c) $\mathbb{N}$ and $\mathbb{Q}$
   (d) $\mathbb{N} \times \{1, 2, 3, 4\}$ and $\mathbb{N}$
   (e) $[0, 1)$ and $(2, 3]$
   (f) $(0, 1)$ and $\mathbb{R}$
2. Show the sets **do not** have the same cardinality
   (a) $\mathbb{N}$ and $\mathbb{R}$

2 Number Theory

3. Let $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$. Show if $a|b$ and $b|c$ then $a|c$.
4. Let $a, b, c, x, y \in \mathbb{Z}$. Show if $a|b$ and $a|c$ then $a|ax + by$.
5. Let $a, b, c, d \in \mathbb{Z}$ with $c \neq 0$. Show if $a|b$ and $c|d$ then $ac|ad + bc$.
6. Prove $3|n^3 - n$ for every integer n.
7. Let $a, b, c \in \mathbb{Z}$. If $a|bc$ and $\gcd(a, b) = 1$ then $a|c$.
8. Let $a, b \in \mathbb{Z}$ and let p be a prime. If $p|ab$ then $p|a$ or $p|b$.
9. Let $a, b, c \in \mathbb{Z}$ where $\gcd(a, b) = 1$. If $a|c$ and $b|c$ then $ab|c$.
10. Let $n \in \mathbb{Z}$ be odd. Prove $n^2 \equiv 1 \mod 4$.
11. Let $n \in \mathbb{Z}$ be odd. Prove $n^2 \equiv 1 \mod 8$.
12. Let $a, b, c \in \mathbb{Z}$ so that $a^2 + b^2 = c^2$. Prove $4|ab$ (hint use 11).
13. Prove or disprove if $a, b \in \mathbb{Z}$ and a, b are odd then $8|a^2 - b^2$.
14. Prove $\sqrt{3}$ is irrational.

15. For the following pairs of numbers, find their gcd and and find a linear combination of the numbers equal to their gcd.

    (a) $a = 78$ and $b = 48$

    (b) $a = 79$ and $b = 49$

    (c) $a = 253$ and $b = 207$

3 Group Theory

16. State the definition of an **Algebraic sturcture** and a **binary operation**.

17. State the definition of a **group**.

18. State the definition of an **Abelian group**.

19. State and prove the cancellation law.

20. Are the following algebraic structures groups? If it is a group prove it. If not prove it is not a group.

    (a) $(\mathbb{Q}^*, \cdot)$ where $\cdot$ is regular multiplication.

    (b) $(\mathbb{Z}, \odot)$ where $a \odot b = a + b - 2$.

    (c) $(\mathbb{Z}, \circledast)$ where $a \circledast b = ab + a + b$.

    (d) $(\mathbb{Q} \setminus \{-1\}, \circledast)$ where $a \circledast b = ab + a + b$.

21. For the following algebraic structures write the operation tables. State if the structure is a group or not. If not state which property fails and how. Identify the identity in each case.

    (a) $(\mathbb{Z}_5, +)$

    (b) $(\mathbb{Z}_5, \cdot)$

    (c) $(\mathbb{Z}_6^*, \cdot)$

    (d) $(S_3, \circ)$

    (e) $(D, \circ)$ where

$$D = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\}$$

(f) $(D, \circ)$ where $D \subseteq S_3$ defined as

$$D = \{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\}$$

(g) $(M, \cdot)$ where $M \subseteq GL_2(\mathbb{R})$ defined as

$$M = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\}$$

22. We showed in class that some groups are "the same" by finding a function from one group to the other that preserved the operation. The following pairs of groups are "the same" in that sense find the function and verify that it preserves the operation.

    (a) $(Z_4, +)$ and $(Z_5^*, \cdot)$

    (b) $(Z_2, +)$ and $(M, \cdot)$ where $M \subseteq GL_2(\mathbb{R})$ defined as

    $$M = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$$

    (c) $(Z_{12}^*, \cdot)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

23. Let G be a group prove

    (a) The identity in G is unique.

    (b) If $a \in G$ then a^{-1} is unique.

    (c) If $a \in G$ then $(a^{-1})^{-1} = a$.

    (d) If $a, b \in G$ then $(ab)^{-1} = b^{-1}a^{-1}$.

24. Let G be a group prove and let $a, b \in G$. If a and b commute then a^{-1} and b^{-1} commute.

25. Let $(G, *)$ be a group prove and let $a, b, c \in G$. Prove the following equations have an unique solution $x \in G$.

    - $a * x * c = b$
    - $c * a * x = b$

26. Let $(G, *)$ be a group prove.
    Assume there is some element $f \in G$ so that for some elemnt $a \in G$ we have $af = a$. Is f the identity? Prove or disprove.

27. Let $(G, *)$ be a group prove.
    If $a * a = e$ for all $a \in G$ then G is Abelian.