

Name: _____

Measure Theory Quiz 2

1. Show the following definitions are equivalent.

- A function $f : A \rightarrow B$ is **continuous** if for all $a \in A$ and for all $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.
- A function $f : A \rightarrow B$ is **continuous** if for all $U \in B$ open, $f^{-1}(U)$ is open in A .

2. Find a sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ so that

- $\int f_n(x) dx = 1$ and
- $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all $x \in [0, 1]$.

3. Compute the lim sup and lim sup of the following.

- $a_n = (-1)^n$
- $a_n = n^4(-1.01)^n$
- $c_n = q_n$ where $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ is an enumeration of the rationals.

4. Show that $\mathbb{Q}$ is a null set. Recall the definition of a null set.

A set $A \subseteq \mathbb{R}$ is a **null set** if for all $\varepsilon > 0$ there is a sequence of intervals $(I_n)_{n=1}^{\infty}$ so that

- $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and
- $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$