5 Some Proofs And Some Answers

19. Prove and state the product rules for the dot product of two paths. There are three product rules for vectors

5.1 Scalar Multiplication

$$\frac{d}{dx}\left[f(t)\mathbf{r}(t)\right] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$$

Proof. Let f(t) be a scalar function and let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ be a vector valued function where $x(t), y(t)$ are scalar functions.

$$\begin{aligned}
\frac{d}{dt}\left[f(t)\mathbf{r}(t)\right] &= \frac{d}{dt}\left[f(t)\langle x(t), y(t)\rangle\right] \\
&= \frac{d}{dt}\left[\langle f(t)x(t), f(t)y(t)\rangle\right] \\
&= \langle f'(t)x(t) + f(t)x'(t), f'(t)y(t) + f(t)y'(t)\rangle \\
&= \langle f'(t)x(t), f'(t)y(t)\rangle + \langle f(t)x'(t), f(t)y'(t)\rangle \\
&= f'(t)\langle x(t), y(t)\rangle + f(t)\langle x'(t), y'(t)\rangle \\
&= f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)
\end{aligned}$$

□

5.2 Dot Product

$$\frac{d}{dx}\left[\mathbf{r_1}(t) \cdot \mathbf{r_2}(t)\right] = \mathbf{r_1}'(t) \cdot \mathbf{r_2}(t) + \mathbf{r_1}(t) \cdot \mathbf{r_2}'(t)$$

Proof. Let $\mathbf{r_1}(t) = \langle x_1(t), y_1(t)\rangle$ and $\mathbf{r_2}(t) = \langle x_2(t), y_2(t)\rangle$ be vector valued functions where $x_1(t), y_1(t), x_2(t), y_2(t)$ are scalar functions.

$$
\begin{aligned}
\frac{d}{dt}\left[\mathbf{r_1}(t)\cdot\mathbf{r_2}(t)\right] &= \frac{d}{dt}\left[\langle x_1(t),y_1(t)\rangle\cdot\langle x_2(t),y_2(t)\rangle\right]\\
&= \frac{d}{dt}\left[x_1(t)x_2(t)+y_1(t)y_2(t)\right]\\
&= x_1'(t)x_2(t)+x_1(t)x_2'(t)+y_1(t)'y_2(t)+y_1(t)y_2'(t) \text{ regular product rule}\\
&= x_1'(t)x_2(t)+y_1(t)'y_2(t)+x_1(t)x_2'(t)+y_1(t)y_2'(t)\\
&= \langle x_1'(t),y_1'(t)\rangle\cdot\langle x_2(t),y_2(t)\rangle+\langle x_1(t),y_1(t)\rangle\cdot\langle x_2'(t),y_2'(t)\rangle\\
&= \mathbf{r_1}'(t)\cdot\mathbf{r_2}(t)+\mathbf{r_1}(t)\cdot\mathbf{r_2}'(t)
\end{aligned}
$$

□

5.3 Cross Product

$$
\frac{d}{dx}\left[\mathbf{r_1}(t)\times\mathbf{r_2}(t)\right]=\mathbf{r_1}'(t)\times\mathbf{r_2}(t)+\mathbf{r_1}(t)\times\mathbf{r_2}'(t)
$$

Proof. Let $\mathbf{r_1}(t)=\langle x_1(t),y_1(t),z_1(t)\rangle$ and $\mathbf{r_2}(t)=\langle x_2(t),y_2(t),z_2(t)\rangle$ be vector valued functions where $x_1(t),y_1(t),z_1(t),x_2(t),y_2(t),z_2(t)$ are scalar functions.

This one is for you to do ... □

20. Prove: If $\|\mathbf{r}(t)\|$ is constant then $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$.

Proof. Let $\mathbf{r}(t)$ be a vector valued function with constant norm. That is $\|\mathbf{r}(t)\|=k$ for some $k\in\mathbb{R}$. We will show $\mathbf{r}(t)\cdot\mathbf{r}'(t)=0$ therefore $\mathbf{r}(t)$ will be perpendicular to $\mathbf{r}'(t)$.

We will calculate $\frac{d}{dt}\left[\|\mathbf{r}(t)\|^2\right]$ two ways and set them equal.

$$
\begin{aligned}
\frac{d}{dt}\left[\|\mathbf{r}(t)\|^2\right] &= \frac{d}{dt}\left[k^2\right] && (1)\\
&= 0 \text{ the derivative of a constant.} && (2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\left[\|\mathbf{r}(t)\|^2\right] &= \frac{d}{dt}\left[\mathbf{r}(t)\cdot\mathbf{r}(t)\right] \text{ since } \|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v} && (3)\\
&= \mathbf{r}'(t)\cdot\mathbf{r}(t)+\mathbf{r}(t)\cdot\mathbf{r}'(t) \text{ product rule} && (4)\\
&= \mathbf{r}'(t)\cdot\mathbf{r}(t)+\mathbf{r}'(t)\cdot\mathbf{r}(t) \text{ commutativity} && (5)\\
&= 2\mathbf{r}'(t)\cdot\mathbf{r}(t) && (6)
\end{aligned}
$$

Thus the resukts above are equal. That is

$$2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$

So $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$.

$\square$