Math 4100 - Test 1

Name:____

1. Calculate $\phi(840)$ and $\phi(841)$ (neither are prime).

2. Find the remainder when

   (a) $2(4294!)$ is divided by 4297 (Hint: Use Wilson's theorem and 4297 is prime).

   (b) 53^{103} is divided by 39.

3. If the $\gcd(a, 42) = 1$ then show $a^6 \equiv 1 \mod 42$.

4. Let

$$104x + 301y = 42 \tag{1}$$

(a) Find one integer solution to equation (1).

(b) Find all integer solutions to equation (1) where x and y are both positive.

5. Solve using the Chinese Remainder Theorem. Find all integer solutions.

$$\begin{aligned} x &= 5 \mod 6 \\ x &= 4 \mod 11 \\ x &= 3 \mod 17 \end{aligned} \tag{2}$$

6. If $a|b$ and $a|c$ then $a^2|bc$.

7. Let

$$26 = x_1 + 2x_2 + 4x_3 + 9x_4 + 20x_5 \tag{3}$$

where we consider solutions of the form $x_i \in \{0, 1\}$. That is the solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ and $x_5 = 1$. So equation 3 becomes

$$26 = (0) + 2(1) + 4(1) + 9(0) + 20(1). \tag{4}$$

You may begin to notice the string $x_1x_2x_3x_4x_5 = 01101$ is starting to resemble a string of ones and zeroes in a computer.

(a) Find a bitwise solution (ie find x_i so that each $x_i \in \{0, 1\}$) to

$$32 = x_1 + 2x_2 + 4x_3 + 9x_4 + 20x_5. \tag{5}$$

(b) Look at the equation

$$35 = x_1 + 3x_2 + 5x_3 + 10x_4 + 21x_5. \tag{6}$$

Now we mulltiply equation 7 by 14 and get

$$490 = 14x_1 + 42x_2 + 70x_3 + 140x_4 + 294x_5. \tag{7}$$

Next we reduce equation 7 modulo 45 and get

$$40 = 14x_1 + 42x_2 + 25x_3 + 5x_4 + 24x_5. \tag{8}$$

Find a bitwise solution (ie find x_i so that each $x_i \in \{0, 1\}$) to equation (8).

(c) Note for (b) there are only $2^5 = 32$ possible choices (of 1's and zeroes). Fairly easy to check all of the possible solutions by computer or even by hand. What if we were to code eight words (a word is four bytes and a byte is eight words, so 32 bits per word and 256 bits for eight words). A relativley modest amount of data in today's computing world. We would need $x_1, x_2, \ldots, x_{256}$.

i. How many possible choices (of 1's and zeroes)are there?

ii. If you had a billion (10^9) computers each so fat the could check a trillion (10^{12}) possibilities per second, How many years would it take to check all possible answers (assume 365 days per year)?

8. Six sailors survive a shipwreck and swim to a tiny island where there is nothing but a coconut tree and a monkey. The sailors gather all the coconuts and put them in a big pile under the tree. Exhausted, they agree to wait until the next morning to divide up the coconuts. At one o'clock in the morning, the first sailor wakes. He realizes that he can't trust the others, and decides to take his share now. He divides the coconuts into six equal piles, but there is one left over. He gives that coconut to the monkey, buries his coconuts, and puts the rest of the coconuts back under the tree. At two o'clock, the second sailor wakes up. Not realizing that the first sailor has already taken his share, he too divides the coconuts up into six piles, leaving one left over which he gives to the monkey. He then hides his share, and piles the remainder back under the tree. At three o'clock, the third sailor wakes up and carries out the same actions. And so do the fourth fifith and sixth sailor.

   Later in the morning, all the sailors wake up, and try to look innocent. No one makes a remark about the diminished pile of coconuts, and no one decides to be honest and admit that they've already taken their share. Instead, they divide the pile up into six piles, for the seventh time, and find that there is yet again one coconut left over, which they give to the monkey.

   How many coconuts were there originally?