Math 4100 Practice Test 1

1. Understand the quizzes plus these problems ...

2. Use the division algorithm to show
   - The square of any number is of the form $3k$ or $3k+1$.
   - The fourth power of any number is of the form $5k$ or $5k+1$.

3. Prove the following where $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$ then $a^2|bc$.

4. Prove for $n \in \mathbb{N}$ (using induction) that
   - $15|2^{4n} - 1$.
   - $5|3^{3n+1} + 2^{n+1}$.

5. The product of four consequetive integers is 1 less than a perfect square.

6. Use the Euclidean algorithm to obtain integer solutions for x and y.
   - $\gcd(119, 272) = 119x + 272y$
   - $\gcd(1769, 2378) = 1769x + 2378y$

7. Deterimine all solutions to the Diophantine equations
   - $18x + 5y = 48$
   - $54x + 21y = 906$
   - $123x + 360y = 99$

8. Solve the following Chinese Remainder Theorem problems.
   - $$\begin{aligned} x &\equiv 3 \mod 5 \\ 2x &\equiv 3 \mod 7 \\ 6x &\equiv 3 \mod 11 \end{aligned}$$
   - Problem 4.4.8

9. Compute $\tau(1001)$, $\tau(5040)$ and $\tau(36000)$. Compute $\sigma(1001)$, $\sigma(5040)$ and $\sigma(36000)$. Compute $\phi(1001)$, $\phi(5040)$ and $\phi(36000)$.

10. Verify $\phi(n) = \phi(n+1) = \phi(n+2)$ where $n = 5186$.

11. Prove if n is odd then $\phi(2n) = \phi(n)$

12. Prove if n is even then $\phi(2n) = 2\phi(n)$

13. Prove $\phi(3n) = 3\phi(n)$ if and only if $3|n$.

14. Use Euler's Theorem to establish

    (a) For $a \in \mathbb{Z}$ with $\gcd(a, 1729) = 1$ we have $a^{36} = 1 \mod 1729$. Hint $1729 = (7)(13)(19)$.

    (b) That $51|10^{32n+9} - 7$.

    (c) For any two relatively prime integers m and n we have $m^{\phi(n)} + n^{\phi(m)} = 1 \mod (mn)$.

    (d) Find the units digits of 3^{100} or the units digits of 3^{407}.

15. Find the order of the integers 2,3 and 5

    (a) modulo 17

    (b) modulo 19

    (c) modulo 23

16. Prove

    (a) If a has order hk modulo n then a^h has order k modulo n.

    (b) If a has order $2k$ modulo p, an odd prime, then $a^k = -1 \mod p$.

    (c) If a has order 3 modulo p, where p is an odd prime then $(a+1)^6 = 1 \mod p$. [Hint note $a^2 + a + 1 = 0 \mod p$ thus $(a+1)^2 = a \mod p$].

17. Plus problems 6.1.7, 7.2.4, 7.2.5, 7.2.10